Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz space, heat kernel

Bruno Schapira

Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz space, heat kernel

Bruno Schapira*

2nd May 2006

Université d'Orléans,
Fédération Denis Poisson, Laboratoire MAPMO
B.P. 6759, 45067 Orléans cedex 2, France .

Université Pierre et Marie Curie,
Laboratoire de Probabilités et Modèles Aléatoires,
4 place Jussieu,
F-75252 Paris cedex 05, France.

Abstract

Under the assumption of positive multiplicity, we obtain basic estimates of the hypergeometric functions F_λ and G_λ of Heckman and Opdam, and sharp estimates of the particular functions F_0 and G_0. Next we prove the Paley-Wiener theorem for the Schwartz class, solve the heat equation and estimate the heat kernel.

Key Words: Differential-difference equations, hypergeometric functions, root systems, Schwartz space, heat kernel.

A.M.S. Classification. *Primary:* 33C67, 33D67, 33E30, 42B10, 58J35. *Secondary:* 35K05, 42A90, 43A32, 47D07, 58J65.

e-mail. bruno.schapira@univ-orleans.fr

*partially supported by the European Commission (IHP Network HARP 2002 – 2006).

1 Introduction

Classical harmonic analysis on $\mathbb{R}^n$ has now been extended to other spaces. For instance Harish-Chandra has considered the case of semi-simple Lie groups. Then he was followed by Helgason, who studied the Riemannian symmetric spaces of noncompact type, which are Riemannian spaces of negative curvature. In particular, Harish-Chandra introduced and studied the spherical functions, which play the role of the exponentials in these spaces. A more general setting, in the flat case, has appeared two or three decades ago, with the theory of Dunkl operators. It gives a vast generalization of the exponential functions, and of the Fourier transform on $\mathbb{R}^n$. But it gives also a generalization of the harmonic analysis on tangent spaces of symmetric spaces. The natural counterpart of the Dunkl theory in the negatively curved setting is the theory of Heckman and Opdam. This theory has known a deep evolution with the discovery of the Cherednik operators [6], the analogues of the Dunkl operators in the flat case.
Heckman and Opdam [11], [12], [14] have developed their theory in the last two decades. They have first introduced a new family of functions F_λ on $\mathbb{R}^n$, which like in the Dunkl theory are associated to root systems and a parameter, the multiplicity function. They can be defined essentially as eigenfunctions of certain differential operators. When the multiplicity function, takes particular values, then these operators coincide with the radial part of the G-invariant differential operators on the symmetric spaces of noncompact type G/K. Thus the restrictions to a Cartan subspace $\mathfrak{a}$ of the spherical functions are particular functions F_λ. In this way the theory of Heckman and Opdam is also a generalization of the harmonic analysis on the symmetric spaces G/K. However all the techniques used by Harish-Chandra can not always be transposed (at least not trivially) in this new theory, because there are not anymore underlying Lie groups. The main tools used in the harmonic analysis on the symmetric spaces are in the one part an integral formula of the spherical functions, and in another part a development in series of these spherical functions. Heckman and Opdam have shown that their functions F_λ have a development in series of the type Harish-Chandra, but there is not (at least not yet) an integral formula, for general root systems. However this gap has been compensated by two main discoveries. First the discovery of the differential-difference operators by Cherednik [6], and then the discovery by Opdam of a new type of functions, the functions G_λ [14], for which the calculus and estimates can be more easily performed. These functions are eigenfunctions of the Cherednik operators. However until recently the only asymptotic result was essentially the fact that the functions F_λ and G_λ were bounded [14]. Delorme has obtained a much better estimate, even in the more complicated case of a negative multiplicity [8], but it requires involved materials and techniques.
In this paper we give sharp estimates of the functions F_λ, G_λ and their derivatives, in an elementary way. Our method is only based on the study of the system of differential and difference equations satisfied by the functions G_λ, improving by the way what had already done De Jeu [13] and Opdam [14] for bounding their functions. We also give a global estimate of the particular

functions F_0 and G_0. It generalizes some results in the noncompact symmetric spaces [1], [3]. Then we deduce from these estimates and from a general method of Anker [2] the inversion formula on the Schwartz space. Finally we solve the Heat equation and we give some estimates of the heat kernel.

Acknowledgments: This work is part of my PhD. It is a great pleasure to thank my advisors Jean-Philippe Anker and Philippe Bougerol for their help and advices.

2 Preliminaries

Let $\mathfrak{a}$ be a Euclidean vector space of dimension n, equipped with an inner product $(\cdot,\cdot)$. Let $\mathfrak{h} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{a}$. The notation $\Re$ and $\Im$ denote the real and imaginary part respectively, of an element in $\mathfrak{h}$ or possibly in $\mathbb{C}$. Let $\mathcal{R} \subset \mathfrak{a}$ be an integral root system. We choose a subset of positive roots $\mathcal{R}^+$. We denote by $\mathcal{R}_0^+$ the set of positive indivisible roots, by Π the set of simple roots, and by Q^+ the positive lattice generated by $\mathcal{R}^+$. Let $\alpha^\vee = \frac{2}{|\alpha|^2}\alpha$ be the coroot associated to a root α and let

$$r_\alpha(x) = x - (\alpha^\vee, x)\alpha,$$

be the corresponding orthogonal reflection. We denote by W the Weyl group associated to $\mathcal{R}$, i.e. the group generated by the r_α's. If C is a subset of $\mathfrak{a}$, we call *symmetric* of C any image of C under the action of W. Let $k : \mathcal{R} \to [0, +\infty)$ be a multiplicity function, which by definition is W-invariant. In the sequel we may actually forget about the roots α with $k_\alpha = 0$ and restrict ourself to the root subsystem where k is strictly positive.
Let

$$\mathfrak{a}_+ = \{x \mid \forall \alpha \in \mathcal{R}^+,\ (\alpha, x) > 0\},$$

be the positive Weyl chamber. We denote by $\overline{\mathfrak{a}_+}$ its closure, and by $\partial\mathfrak{a}_+$ its boundary. Let also $\mathfrak{a}_{\mathrm{reg}}$ be the subset of regular elements in $\mathfrak{a}$, i.e. those elements which belong to no hyperplane $\{\alpha = 0\}$. For I a subset of $\mathcal{R}^+$, let

$$\mathfrak{a}^I := \{x \in \mathfrak{a} \mid \forall \alpha \in I,\ (\alpha, x) = 0\}$$

be the face associated to I. Let $\mathcal{R}_I$ be the set of positive roots which are orthogonal to $\mathfrak{a}^I$, and let W_I be the subgroup of W generated by the r_α with $\alpha \in \mathcal{R}_I$.
For $\xi \in \mathfrak{a}$, let T_ξ be the Dunkl-Cherednik operator. It is defined, for $f \in C^1(\mathfrak{a})$, and $x \in \mathfrak{a}_{\mathrm{reg}}$, by

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \frac{(\alpha, \xi)}{1 - e^{-(\alpha, x)}}\{f(x) - f(r_\alpha x)\} - (\rho, \xi) f(x),$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+} k_\alpha \alpha.$$

The Dunkl-Cherednik operators form a commutative family of differential-difference operators (see [6] or [14]). The Heckman-Opdam Laplacian $\mathcal{L}$ is defined by

$$\mathcal{L} = \sum_{i=1}^{n} T_{\xi_i}^2,$$

where $\{\xi_1, \ldots, \xi_n\}$ is any orthonormal basis of $\mathfrak{a}$ ($\mathcal{L}$ is independent of the chosen basis). Here is an explicit expression (see the appendix), which holds for $f \in C^2(\mathfrak{a})$ and $x \in \mathfrak{a}_{\text{reg}}$:

$$\begin{aligned} \mathcal{L}f(x) &= \Delta f(x) + \sum_{\alpha\in\mathcal{R}^+} k_\alpha \coth\frac{(\alpha,x)}{2}\partial_\alpha f(x) + |\rho|^2 f(x) \qquad (1)\\ &- \sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{|\alpha|^2}{4\sinh^2\frac{(\alpha,x)}{2}}\{f(x) - f(r_\alpha x)\} \end{aligned}$$

Let $\lambda \in \mathfrak{h}$. We denote by F_λ the unique analytic W-invariant function on $\mathfrak{a}$, which satisfies the differential equations

$$p(T_\xi)F_\lambda = p(\lambda)F_\lambda \text{ for all W-invariant polynomials } p$$

and which is normalized by $F_\lambda(0) = 1$ (in particular $\mathcal{L}F_\lambda = (\lambda, \lambda)F_\lambda$). We denote by G_λ the unique analytic function on $\mathfrak{a}$, which satisfies the differential and difference equations

$$T_\xi G_\lambda = (\lambda, \xi)G_\lambda \text{ for all } \xi \in \mathfrak{a}, \qquad (2)$$

and which is normalized by $G_\lambda(0) = 1$.

The $\mathbf{c}$*-function.*
We define the function $\mathbf{c}$ as follows (see [10] or [11]):

$$\mathbf{c}(\lambda) = c_0 \prod_{\alpha\in\mathcal{R}^+} \frac{\Gamma(-(\lambda,\alpha^\vee) + \frac{1}{2}k_{\frac{\alpha}{2}})}{\Gamma(-(\lambda,\alpha^\vee) + k_\alpha + \frac{1}{2}k_{\frac{\alpha}{2}})},$$

where c_0 is a positive constant chosen in such a way that $\mathbf{c}(-\rho) = 1$, and $k_{\frac{\alpha}{2}} = 0$ if $\frac{\alpha}{2} \notin \mathcal{R}$. Observe that if

$$\pi(\lambda) := \prod_{\alpha\in\mathcal{R}_0^+} (\lambda, \alpha^\vee),$$

then the function

$$\mathbf{b}(\lambda) := \pi(\lambda)\mathbf{c}(\lambda),$$

is analytic in a neighborhood of 0.

Remark 2.1 For the reader's convenience, let us point out a conventional difference between our setting and symmetric spaces. There Σ denotes the root system and $m : \Sigma \to \mathbb{N}^*$ the multiplicity function. Everything fits together if we set $\mathcal{R} = 2\Sigma$ and $k_{2\alpha} = \frac{1}{2} m_\alpha$. Notice in particular that ρ is defined in the same way in both settings:

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha = \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+} k_\alpha \alpha.$$

3 Estimates

3.1 Positivity and first estimates

Let us begin with the following positivity result.

Lemma 3.1 *Assume that $\lambda \in \mathfrak{a}$. Then the functions F_λ and G_λ are real and strictly positive.*

Proof of lemma: Since

$$F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(w \cdot x), \; x \in \mathfrak{a}, \tag{3}$$

it is enough to prove the lemma for G_λ. First of all, the function G_λ is real valued, since G_λ and $\overline{G_\lambda}$ satisfy the same equations (2), and hence are equal. Assume next that G_λ vanishes. Let x be a zero of G_λ of minimal norm $r = |x|$. Consider first the case where x is a regular point, and take a vector ξ in the same chamber as x. As G_λ is positive for $|x| < r$, we have

$$\partial_\xi G_\lambda(x) \leq 0.$$

Writing down (2), we get

$$\partial_\xi G_\lambda(x) = \sum_{\alpha \in \mathcal{R}^+} k_\alpha \frac{(\alpha, \xi)}{1 - e^{-(\alpha,x)}} (G_\lambda(r_\alpha x) - G_\lambda(x)) + (\rho + \lambda, \xi) G_\lambda(x). \tag{4}$$

Since for all roots α,

$$\frac{(\alpha, \xi)}{1 - e^{-(\alpha,x)}} \geq 0,$$

we deduce that $\partial_\xi G_\lambda(x) = 0$, and that $G_\lambda(r_\alpha x) = 0$ for every $\alpha \in \mathcal{R}$. Hence G_λ and ∇G_λ vanish at the point x and furthermore at each conjugate of x under W. Differentiating (4), we see that every second order partial derivative of G_λ vanishes on the W-orbit of x. And similarly for all higher order derivatives. Since G_λ is analytic, we deduce that $G_\lambda \equiv 0$. This contradicts the fact that $G_\lambda(0) = 1$.

Consider next the case where x is singular and let $I = \{\alpha \in \mathcal{R}^+ \mid (\alpha, x) = 0\}$. The equations (2) become now

$$\begin{aligned} \partial_\xi G_\lambda(x) &= -\sum_{\alpha \in I} 2k_\alpha \frac{(\alpha, \xi)}{|\alpha|^2} \partial_\alpha G_\lambda(x) \\ &+ \sum_{\alpha \in \mathcal{R}^+ \setminus I} k_\alpha \frac{(\alpha, \xi)}{1 - e^{-(\alpha,x)}} (G_\lambda(r_\alpha x) - G_\lambda(x)) + (\rho + \lambda, \xi) G_\lambda(x). \end{aligned} \tag{5}$$

We may argue as before, taking $\xi \in \mathfrak{a}^I$ in the same face as x. Notice that the first sum vanishes in the right hand side of (5), and that

$$\partial_\xi (r_\alpha G_\lambda)(x) = \partial_{r_\alpha \xi} G_\lambda(r_\alpha x).$$

with $r_\alpha \xi$ in the same face as $r_\alpha x$. Eventually we obtain that all partial derivatives of G_λ along directions belonging to $\mathfrak{a}^I$ vanish at x. Again since G_λ is analytic, it must vanish on $\mathfrak{a}^I$, which contradicts $G_\lambda(0) = 1$. This concludes the proof of the lemma. ∎

The next proposition is fundamental in order to have uniform estimates in the parameter $\lambda \in \mathfrak{h}$.

Proposition 3.1 *(a) For all $\lambda \in \mathfrak{h}$,*

$$|G_\lambda| \leq G_{\Re(\lambda)}.$$

(b) For all $\lambda \in \mathfrak{a}$ and for all $x \in \mathfrak{a}$

$$G_\lambda(x) \leq G_0(x) e^{\max_w (w\lambda, x)}.$$

Proof of the proposition: For the first inequality, we study the behavior of the ratio $Q_\lambda = \frac{G_\lambda}{G_{\Re(\lambda)}}$. We must show that $|Q_\lambda|^2 \leq 1$. We will in fact prove that for all $\xi \in \mathfrak{a}_{\mathrm{reg}}$,

$$M(\xi, r) := \max_{w \in W} |Q_\lambda(rw\xi)|^2$$

is a decreasing function of $r \geq 0$. Since $M(\xi, 0) = 1$ for all ξ, the result will follow. First of all observe that the function M is continuous and right differentiable in the second variable r. Then, using (2), we get

$$\partial_\xi |Q_\lambda|^2(x) = \sum_{\alpha \in \mathcal{R}^+} \frac{2k_\alpha(\alpha, \xi)}{1 - e^{-(\alpha, x)}} (\Re\{Q_\lambda(x)\overline{Q_\lambda}(r_\alpha x)\} - |Q_\lambda(x)|^2) \frac{G_{\Re(\lambda)}(r_\alpha x)}{G_{\Re(\lambda)}(x)},$$

for all ξ and all x regular. Hence if x is a regular element such that

$$|Q_\lambda(x)|^2 = \max_w |Q_\lambda(wx)|^2,$$

and if ξ is a positive multiple of x, we have

$$\partial_\xi |Q_\lambda|^2(x) \leq 0.$$

This means that

$$\frac{\partial M}{\partial r}(\xi, |x|) \leq 0,$$

where we consider right derivatives. So for every ξ regular, and every $r \geq 0$,

$$\frac{\partial M}{\partial r}(\xi, r) \leq 0.$$

In order to conclude, we need the following elementary lemma, whose proof is left to the reader.

Lemma 3.2 *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous and right derivable function. We denote by f'_d the right derivative of f. If for all $x \in \mathbb{R}^+$, $f'_d(x) \leq 0$, then f is decreasing.*

According to this lemma, we have $M(\xi, r) \leq M(\xi, 0) = 1$, for all $\xi \in \mathfrak{a}_{\text{reg}}$ and all $r \geq 0$. By continuity, this inequality remains true if ξ is singular. This concludes the proof of the first inequality.
The second one is proved similarly, using the ratio

$$R_\lambda(x) := \frac{G_\lambda(x) e^{-\max_w (w\lambda, x)}}{G_0(x)}.$$

Specifically, if x is regular and $\xi \in \mathfrak{a}$, then

$$\begin{aligned} \partial_\xi R_\lambda(x) &= \sum_{\alpha \in \mathcal{R}^+} \frac{k_\alpha (\alpha, \xi)}{1 - e^{-(\alpha, x)}} (R_\lambda(r_\alpha x) - R_\lambda(x)) \frac{G_0(r_\alpha x)}{G_0(x)} \\ &+ ((\lambda, \xi) - \max_w (w\lambda, \xi)) R_\lambda(x), \end{aligned}$$

where we consider again right derivatives. So if x is such that

$$R_\lambda(x) = \max_w R_\lambda(wx)$$

and ξ is a positive multiple of x, then

$$\partial_\xi R_\lambda(x) \leq 0.$$

Therefore

$$N(\xi, r) := \max_{w \in W} R_\lambda(rw \cdot \xi)$$

is a decreasing function in $r \geq 0$, for all $\xi \in \mathfrak{a}_{\text{reg}}$. We conclude as for the first inequality. ∎

By averaging over the Weyl group, we deduce the following inequalities from Proposition 3.1.

Corollary 3.1 *1. For all $\lambda \in \mathfrak{h}$,*

$$|F_\lambda| \leq F_{\Re(\lambda)}.$$

2. For all $\lambda \in \mathfrak{a}$ and for all $x \in \mathfrak{a}$

$$F_\lambda(x) \leq F_0(x) e^{\max_{w \in W} (w\lambda, x)}.$$

3.2 Local Harnack principles and sharp global estimates

In this subsection we first establish two Harnack principles for G_λ and F_λ when $\lambda \in \mathfrak{a}$, and next deduce sharp global estimates of these functions F_λ and of the function G_0. Before stating the results we introduce some new notation. Let I

be a subset of $\mathcal{R}^+$, and let $d \leq d'$ be two strictly positive constants. We denote by $V^I(d,d')$ the following subset of $\mathfrak{a}$:

$$V^I(d,d') := \{x \in \mathfrak{a} \mid \forall \alpha \in \mathcal{R}_I,\ |(\alpha,x)| \leq d \text{ and } \forall \alpha \notin \mathcal{R}_I,\ |(\alpha,x)| > d'\}.$$

Let $x \in V^I(d,d')$, with I non empty. Let $p^I(x)$ denote its orthogonal projection on $\mathfrak{a}^I$. Let $u \in \mathfrak{a}^I$ be such that for every $\alpha \notin \mathcal{R}_I$, $(\alpha,u)\mathrm{sgn}((\alpha,x)) \geq |\alpha|$. Define now the vectors $\xi_1(x)$, and $\eta_1(x)$ as follows:

$$\xi_1(x) = \frac{p^I(x)-x}{|p^I(x)-x|} + u, \text{ and } \eta_1(x) = \frac{p^I(x)-x}{|p^I(x)-x|} - u.$$

We will sometime just write them ξ_1 and η_1 for simplify the notation. Notice that everything was done in order that

$$\forall \alpha \notin \mathcal{R}_I,\ (\alpha,\xi_1(x))(\alpha,x) \geq 0 \text{ and } (\alpha,\eta_1(x))(\alpha,x) \leq 0. \tag{6}$$

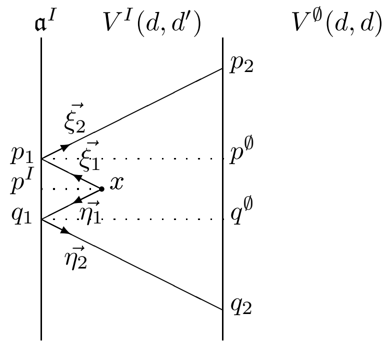
Naturally we have also

$$\forall \alpha \in \mathcal{R}_I,\ (\alpha,\xi_1(x))(\alpha,x) = (\alpha,\eta_1(x))(\alpha,x) \leq 0. \tag{7}$$

We denote by p_1 and q_1 the projections of x on $\mathfrak{a}^I$ along the directions ξ_1 and η_1 respectively (we suppose that d' is sufficiently large in order that these projections still lie in the same chamber than x). Then we denote by $p^\emptyset$ and $q^\emptyset$ the orthogonal projections of p_1 and q_1 respectively on $V^\emptyset(d,d)$. We define also the vectors ξ_2 and η_2 (like before we forget the dependence in x in the notation) by

$$\xi_2 = \frac{p^\emptyset - p_1}{|p^\emptyset - p_1|} + u, \text{ and } \eta_2 = \frac{q^\emptyset - q_1}{|q^\emptyset - q_1|} - u.$$

Eventually let p_2 and q_2 be the projections on $V^\emptyset(d,d)$ of p_1 and q_1 respectively along the directions ξ_2 and η_2 (here again we suppose that d' is sufficiently large in order that these projections lie in the same chamber than x). We summarize these definitions in the following figure



We can now state the lemma

Lemma 3.3 (Local Harnack principle 1) *Let $\lambda \in \mathfrak{a}$, and let d and d' be chosen as above. There exist two constants $C > 0$ and $c > 0$ such that for all $x \in V^I(d, d')$,*

$$\max_{w \in W_I} G_\lambda(wx) \leq C \min_{w \in W_I} G_\lambda(wp_2(x)),$$

and

$$\min_{w \in W_I} G_\lambda(wx) \geq c \max_{w \in W_I} G_\lambda(wq_2(x)).$$

Proof of the lemma: We begin by the first inequality. Let $x \in V^I$. First remark that $|x - p_1(x)|$ and $|x - q_1(x)|$ are bounded by a constant, say h, which depends only on d. We introduce the function M_λ defined on $\mathfrak{a}$ by:

$$M_\lambda(x) = \max_{w \in W_I} G_\lambda(wx).$$

Let y be such that $G_\lambda(y) = M_\lambda(y)$. We have

$$\begin{aligned} \partial_{\xi_1} G_\lambda(y) &= \sum_{\alpha \in \mathcal{R}_I} k_\alpha \frac{(\alpha, \xi_1)}{1 - e^{-(\alpha, y)}} (G_\lambda(r_\alpha y) - G_\lambda(y)) \\ &+ \sum_{\alpha \in \mathcal{R}^+ \smallsetminus \mathcal{R}_I} k_\alpha \frac{(\alpha, \xi_1)}{1 - e^{-(\alpha, y)}} (G_\lambda(r_\alpha y) - G_\lambda(y)) \\ &+ (\rho + \lambda, \xi_1) G_\lambda(y) \\ &\geq - \sum_{\alpha \in \mathcal{R}^+ \smallsetminus \mathcal{R}_I} k_\alpha \frac{(\alpha, \xi_1)}{1 - e^{-(\alpha, y)}} G_\lambda(y) + (\rho + \lambda, \xi_1) G_\lambda(y). \end{aligned}$$

The lower bound is deduced from our choice of y and from the properties of ξ_1 (6) and (7). Now when $\alpha \in \mathcal{R}^+ \smallsetminus \mathcal{R}_I$, the ratio $\frac{(\alpha, \xi_1)}{1 - e^{-(\alpha, y)}}$ is bounded by a constant which depends only on d'. Thus we can find a constant K, which depends only on d' and λ such that for all $y \in V^I(d, d')$,

$$\partial_{\xi_1} M_\lambda(y) \geq -K M_\lambda(y).$$

Here like in the proof of Proposition 3.1, we consider the right derivatives. Still by Lemma 3.2, we get

$$M_\lambda(x) \leq e^{Kh} M_\lambda(p_1(x)). \tag{8}$$

Now we introduce the function N_λ defined on $\mathfrak{a}$ by

$$N_\lambda(x) = \min_{w \in W_I} G_\lambda(wx).$$

Observe already that N_λ and M_λ are equal on $\mathfrak{a}^I$, and in particular in $p_1(x)$. Moreover, by the same technique as above, we can find a strictly positive constant K' such that

$$N_\lambda(p_1(x)) \leq e^{K'h} N_\lambda(p_2(x)).$$

Together with (8) this proves the first inequality of the lemma. The second one can be proved exactly in the same way, by using this time the intermediate point $q_1(x)$. ∎

We could deduce from this lemma a local Harnack principle for F_λ too. We will instead give a simple expression of the gradient of F_λ, which implies such a principle. Moreover this expression will be needed in the proof of Theorem 3.3.

Lemma 3.4 (Local Harnack principle 2) *For all $x \in \overline{\mathfrak{a}_+}$ and for all $\lambda \in \mathfrak{a}$,*

$$\nabla F_\lambda(x) = -\frac{1}{|W|}\sum_{w\in W} w^{-1}(\rho-\lambda)\ G_\lambda(wx). \tag{9}$$

In particular,

$$|\nabla F_\lambda(x)| \le (|\rho| + |\lambda|)F_\lambda(x).$$

Proof of the lemma: By differentiating (3) we get as above

$$\partial_\xi F_\lambda(x) = \frac{1}{|W|}\sum_{w\in W}\partial_{w\xi}G_\lambda(wx),$$

for all $\xi \in \mathfrak{a}$. Now we use the equations (2), which gives

$$\begin{aligned}
\partial_\xi F_\lambda(x) &= \frac{1}{|W|}\sum_{w\in W}\sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{(\alpha,w\xi)}{1-e^{-(\alpha,wx)}}\{G_\lambda(r_\alpha wx) - G_\lambda(wx)\} \\
&+ \frac{1}{|W|}\sum_{w\in W}(\rho+\lambda, w\xi)G_\lambda(wx) \\
&= -\frac{1}{|W|}\sum_{w\in W}\sum_{\alpha\in\mathcal{R}^+} k_\alpha(\alpha,w\xi)\underbrace{\{\frac{1}{1-e^{-(\alpha,w\xi)}} + \frac{1}{1-e^{(\alpha,w\xi)}}\}}_{=1}G_\lambda(wx) \\
&+ \frac{1}{|W|}\sum_{w\in W}(\rho+\lambda, w\xi)G_\lambda(wx) \\
&= \frac{1}{|W|}\sum_{w\in W}(\lambda-\rho, w\xi)G_\lambda(wx).
\end{aligned}$$

This proves the first claim of the lemma. The second one is an easy consequence, using again (3) and the positivity of G_λ. ∎

We can now deduce a sharp global estimate of F_0 which extends the result of Anker [1] to any multiplicities $k > 0$. Recently Sawyer [18] has obtained the same result for root systems of type A, using explicit formulas.

Theorem 3.1 *In $\overline{\mathfrak{a}_+}$,*

$$F_0(x) \asymp e^{-(\rho,x)}\prod_{\alpha\in\mathcal{R}_0^+}(1+(\alpha,x)).$$

Proof of the theorem: We resume the proof in [1], that we sketch. The local Harnack principle for F_0 (which was deduced in [1] from Harish-Chandra's integral formula) allows us to move the estimate away from the walls in $\mathfrak{a}_+$. There we expand F_0, using the Harish-Chandra series

$$F_\lambda(x) = \sum_{w\in W} \sum_{q\in Q^+} \mathbf{c}(w\lambda)\Gamma_q(w\lambda)e^{(w\lambda-\rho-q,x)}$$

that we multiply by $\pi(\lambda)$ in order to remove the singularity of the $\mathbf{c}$-function at the origin. Then we differentiate with respect to $\pi(\frac{\partial}{\partial\lambda})|_{\lambda=0}$, in order to recover F_0, up to a positive constant. As a result we obtain a converging series

$$F_0(x) = \sum_{q\in Q^+} F_q(x)e^{-(\rho+q,x)},$$

with polynomial coefficients F_q and leading term

$$F_0 e^{-(\rho,x)} \sim \text{const.}\pi(x)e^{-(\rho,x)}.$$

$\blacksquare$

Remark 3.1 We may estimate in a similar way the function F_λ when λ is real. The result reads as follows: for any $\lambda \in \overline{\mathfrak{a}_+}$,

$$F_\lambda(x) \asymp \prod_{\alpha\in\mathcal{R}_0^+ | (\alpha,\lambda)=0} (1+(\alpha,x))e^{(\lambda-\rho,x)}$$

on $\overline{\mathfrak{a}_+}$.

Let us turn to the function G_0. For $x \in \mathfrak{a}$, we denote by x^+ its unique symmetric in $\overline{\mathfrak{a}_+}$.

Theorem 3.2 *In $\mathfrak{a}$,*

$$G_0(x) \asymp \prod_{\alpha\in\mathcal{R}_0^+ | (\alpha,x)\geq 0} (1+(\alpha,x))e^{(-\rho,x^+)}. \tag{10}$$

Proof of the theorem: Let us first show that G_λ has a series expansion in each chamber, like it was done by Opdam in the negative chamber $\mathfrak{a}_-$ [14]. We resume his proof. He first obtained that there exists a polynomial p such that for all $x \in \mathfrak{a}_{\text{reg}}$,

$$(\prod_{\alpha\in\mathcal{R}_0^+} (\lambda,\alpha^\vee) - k_\alpha - 2k_{2\alpha})G_\lambda(x) = p(\lambda, T_\xi)F_\lambda(x).$$

By expanding F_λ and $T = T_\xi$ in each chamber, we find developments of the function G_λ:

$$G_\lambda(x) = \sum_{w'\in W} \mathbf{c}(w^{-1}w'\lambda) \sum_{q\in wQ^+} G_{\lambda,q}^{w,w'} e^{(w'\lambda-w\rho-q,x)}$$

for all $x \in w\mathfrak{a}_+$. Moreover Opdam has proved that $G_{\lambda,0}^{w_0,w'}$ is equal to $|W|\delta_{1,w'}\pi(\lambda)$, where w_0 denotes the longest element in W. Now we apply the same technique as in Theorem 3.1. First we multiply these developments by $\pi(\lambda)$, and then we differentiate with respect to $\pi(\frac{\partial}{\partial\lambda})|_{\lambda=0}$. We get developments of the function G_0 in each chamber:

$$G_0(x) = \sum_{q \in wQ^+} G_q^w(x) e^{-(w\rho+q,x)} \tag{11}$$

for all $x \in w\mathfrak{a}_+$, where the G_q^w are real polynomials. Moreover according to the above mentioned result of Opdam, we see that $G_0^{w_0}$ is a strictly positive constant. Recall some basic notation. The length $l(w)$ of an element of W is defined by

$$l(w) = |\mathcal{R}_0^+ \cap w\mathcal{R}_0^-|.$$

Recall that Π denotes the set of simple roots in $\mathcal{R}^+$. Each $q \in Q^+$ writes $q = \sum_{\alpha\in\Pi} n_\alpha \alpha$, with $n_\alpha \in \mathbb{N}$. We denote by $|q| := \sum_{\alpha\in\Pi} n_\alpha$ the length of q. For $q' \in Q^+$, we write $q' \leq q$, if $q - q' \in Q^+$. Naturally we have similar definitions on wQ^+, where we denote by $|q|_w$ the length of any $q \in wQ^+$ and we write $q' \leq_w q$, if $q' \in wQ^+$ and $q - q' \in wQ^+$. Consider the polynomials

$$\pi_w(x) = \prod_{\alpha \in \mathcal{R}_0^+ \cap w\mathcal{R}_0^+} (\alpha^\vee, x) \quad \text{and} \quad \tilde{\pi}_w(x) = \prod_{\alpha \in \mathcal{R}_0^+ \cap w\mathcal{R}_0^+} \Big(1 + (\alpha^\vee, x)\Big).$$

We need the following lemma, which will be used throughout the proof of Theorem 3.2.

Lemma 3.5 *Let $w \in W$.*

1. *If $\alpha \in \Pi \cap w\mathcal{R}^+$, then $\pi_{r_\alpha w}(r_\alpha x) = \frac{\pi_w(x)}{(\alpha^\vee,x)}$, for all $x \in \mathfrak{a}_{reg}$.*
2. *If $\alpha \in \mathcal{R}_0^+ \cap w\mathcal{R}_0^+$, then $\tilde{\pi}_{r_\alpha w}(r_\alpha x) \leq \frac{\tilde{\pi}_w(x)}{1+(\alpha^\vee,x)}$, for all $x \in w\mathfrak{a}_+$.*
3. *If $\alpha \in \mathcal{R}_0^- \cap w\mathcal{R}_0^+$, then there exists a constant $C > 0$, such that $\tilde{\pi}_{r_\alpha w}(r_\alpha x) \leq C\tilde{\pi}_w(x)(1+(\alpha^\vee,x))^{|\mathcal{R}^+|}$, for all $x \in w\mathfrak{a}_+$.*

Proof of the lemma: Let us prove the first claim. Since $\alpha \in \Pi$, r_α maps $\mathcal{R}_0^+ \smallsetminus \{\alpha\}$ onto itself, hence $\mathcal{R}_0^+ \cap r_\alpha w\mathcal{R}_0^+$ onto $(\mathcal{R}_0^+ \cap w\mathcal{R}_0^+) \smallsetminus \{\alpha\}$. The first claim follows.
Let us prove the second claim. We define therefore an injective map i from $\mathcal{R}_0^+ \cap r_\alpha w\mathcal{R}_0^+$ into $(\mathcal{R}_0^+ \cap w\mathcal{R}_0^+) \smallsetminus \{\alpha\}$, such that $r_\alpha\beta \leq_w i(\beta)$ for all β. The second claim will follow. Let $\beta \in \mathcal{R}_0^+ \cap r_\alpha w\mathcal{R}_0^+$. If $r_\alpha\beta \in \mathcal{R}_0^+$, then we set $i(\beta) = r_\alpha\beta$. Otherwise, we have $(\alpha,\beta) \geq 0$. Hence $r_\alpha\beta \leq_w \beta$. But $r_\alpha\beta \in w\mathcal{R}_0^+$, and therefore $r_\alpha\beta \geq_w 0$. Thus $\beta \in \mathcal{R}_0^+ \cap w\mathcal{R}_0^+$ and we set $i(\beta) = \beta$. The map i defined this way has all required properties.
Let us prove the third claim. We define this time an injective map i from $I \subset \mathcal{R}_0^+ \cap r_\alpha w\mathcal{R}_0^+$ into $\mathcal{R}_0^+ \cap w\mathcal{R}_0^+$ such that, if $\beta \in I$, then $r_\alpha\beta \leq_w i(\beta) + |(\alpha^\vee,\beta)|\alpha$, and otherwise $r_\alpha\beta \leq_w |(\alpha^\vee,\beta)|\alpha$. The third claim will follow. Assume that

$\beta \in \mathcal{R}_0^+ \cap r_\alpha w\mathcal{R}_0^+$. If $r_\alpha\beta \in \mathcal{R}_0^+$, then we set $i(\beta) = r_\alpha\beta$. Otherwise $(\alpha, \beta) \le 0$. Next, either $\beta \in w\mathcal{R}_0^+$, in which case $r_\alpha\beta \le_w \beta + |(\alpha^\vee, \beta)|\alpha$, and we set $i(\beta) = \beta$. Or $\beta \in w\mathcal{R}_0^-$ in which case $r_\alpha\beta \le_w |(\alpha^\vee, \beta)|\alpha$. The map i defined this way has all required properties. ■

By expanding G_0 in (4) according to (11) we get

$$\begin{aligned} \nabla G_q^w(x) &= G_q^w(x)q + \sum_{\alpha\in\mathcal{R}^+\cap w\mathcal{R}^+} k_\alpha G_{r_\alpha q}^{r_\alpha w}(r_\alpha x)\alpha \\ &+ \sum_{\alpha\in w\mathcal{R}^+} k_\alpha \sum_{j\in\mathbb{N}^*} \{G_{r_\alpha(q-j\alpha)}^{r_\alpha w}(r_\alpha x) - G_{(q-j\alpha)}^w(x)\}\alpha, \end{aligned} \tag{12}$$

for all $w \in W$, all $q \in wQ^+$, and all $x \in w\overline{\mathfrak{a}_+}$.

Step 1*:* Let us first establish the estimate

$$|G_0^w(x)| \le C\tilde{\pi}_w(x) \quad \forall w \in W, \quad \forall x \in w\overline{\mathfrak{a}_+}.$$

It is obvious for $w = w_0$. Let us prove it by induction on $l(w_0) - l(w)$. For $q = 0$, (12) amounts to

$$\partial_\xi G_0^w(x) = \sum_{\alpha\in\mathcal{R}^+\cap w\mathcal{R}^+} k_\alpha(\alpha, \xi)G_0^{r_\alpha w}(r_\alpha x).$$

Using the induction hypothesis and Lemma 3.5, we get

$$\begin{aligned} \partial_\xi G_0^w(x) &\le C \sum_{\alpha\in\mathcal{R}^+\cap w\mathcal{R}^+} k_\alpha(\alpha, \xi)\tilde{\pi}_{r_\alpha w}(r_\alpha x) \\ &\le C \sum_{\alpha\in\mathcal{R}^+\cap w\mathcal{R}^+} k_\alpha \frac{(\alpha, \xi)}{1 + (\alpha^\vee, x)}\tilde{\pi}_w(x) \\ &= C\partial_\xi\tilde{\pi}_w(x) \end{aligned}$$

for all $x \in w\overline{\mathfrak{a}_+}$ and $\xi \in w\overline{\mathfrak{a}_+}$, in particular for $\xi \in \mathbb{R}^+x$. Since $G_0^w(0) \le C$ provided C is large enough, we obtain the upper estimate

$$G_0^w(x) \le C\tilde{\pi}_w(x) \quad \forall x \in w\overline{\mathfrak{a}_+}.$$

The same argument yields the lower estimate

$$G_0^w(x) \ge -C\tilde{\pi}_w(x) \quad \forall x \in w\overline{\mathfrak{a}_+}.$$

Step 2*:* Let us next establish the following estimate: There exist a constant $C > 0$ and $h \in \mathfrak{a}_+$, such that for every $w \in W$, $q \in wQ^+$ and $x \in C_h^w := wh + w\overline{\mathfrak{a}_+}$,

$$|G_q^w(x)| \le C^{|q|_w}\tilde{\pi}_w(x)(1 + q(x))^{|\mathcal{R}^+|}. \tag{13}$$

The case $q = 0$ was considered in step 1. Let $q \in Q^+ \smallsetminus \{0\}$ and $w \in W$. Assume that (13) holds for all $(q', w') \in w'Q^+ \times W$ such that $|q'|_w < |q|_w$ or such that

$|q'|_w = |q|_w$ and $l(w') < l(w)$. Using (12), the induction hypothesis and Lemma 3.5, we get

$$\partial_\xi \Big[C^{|q|_w}\tilde{\pi}_w(1+q)^{|\mathcal{R}^+|} - G_q^w\Big](x) \geq (q,\xi)\Big[|\mathcal{R}^+|C^{|q|_w}\tilde{\pi}_w(1+q)^{|\mathcal{R}^+|-1} - G_q^w\Big](x), \quad (14)$$

for all $\xi \in w\mathfrak{a}_+$ and all $x \in wh + w\mathfrak{a}_+$, provided $C > 0$ is large enough. Using now (11) at the point wh we can also assume, by taking again larger C if necessary, that

$$G_q^w(wh) \leq C^{|q|_w},$$

for all $q \in wQ^+$. Let now $u \in wh + w\overline{\mathfrak{a}_+}$ be such that $(1 + (q,u)) = |\mathcal{R}^+|$. Equation (14) implies that

$$[C^{|q|_w}\tilde{\pi}_w(1+q)^{|\mathcal{R}^+|} - G_q^w](x) \geq 0, \quad (15)$$

for all x in the segment $[wh, u]$. For $x = wh + t(u - wh)$ with $t \geq 1$, we have

$$\partial_u \Big[C^{|q|_w}\tilde{\pi}_w(1+q)^{|\mathcal{R}^+} - G_q^w\Big](x) \geq (q,u)\frac{|\mathcal{R}^+|}{(1+(q,x))}\Big[C^{|q|_w}\tilde{\pi}_w(1+q)^{|\mathcal{R}^+|} - G_q^w \vee 0\Big](x).$$

Thus (15) holds also for $x = wh + t(u - wh)$ with $t \geq 1$. This proves the upper estimate

$$G_q^w(x) \leq C^{|q|_w}\tilde{\pi}_w(x)(1+q(x))^{|\mathcal{R}^+|},$$

in C_h^w. The same argument gives the lower estimate

$$G_q^w(x) \geq -C^{|q|_w}\tilde{\pi}_w(x)(1+q(x))^{|\mathcal{R}^+|}.$$

Step 3*:* Let us now find a lower bound for G_0^w. We prove by induction on $l(w_0) - l(w)$ that there exist a constant $c > 0$ and $h \in \mathfrak{a}_+$, such that

$$G_0^w(x) \geq c\pi_w(x)$$

for all $x \in C_h^w$. We suppose that it is true for w' such that $l(w') > l$ and we consider w of length l. By the induction hypothesis there exists some $h \in \mathfrak{a}_+$ and $c > 0$ such that, $G_0^{r_\alpha w}(r_\alpha x) \geq c\pi_{r_\alpha w}(r_\alpha x)$, for all $x \in C_h^w$ and all $\alpha \in \mathcal{R}^+ \cap w\mathcal{R}^+$. Let now $c' > 0$ be another constant. Assume that for some $x_0 \in C_h^w$,

$$[G_0^w - c'\pi_w](x_0) \leq [G_0^w - c'\pi_w](wh) - 1,$$

and suppose that x_0 is such element of minimal norm in C_h^w. Let $(\alpha^*)_{\alpha \in w\Pi}$ be the dual basis of $w\Pi$, i.e. for α and β in $w\Pi$, $(\alpha^*, \beta) = 0$ if $\alpha \neq \beta$ and $= 1$ otherwise. Let $\alpha_0 \in w\Pi$ be such that $(\alpha_0, x_0 - h) = \max_{\beta \in w\Pi}(\beta, x_0 - h)$. It implies that, for small $\epsilon > 0$ at least, $x_0 - \epsilon\alpha_0^* \in C_h^w$. Hence

$$\partial_{\alpha_0^*}[G_0^w - c'\pi_w](x_0) \leq 0.$$

On the other hand we know that for $x \in w\mathfrak{a}_+$,

$$\nabla[G_0^w - c'\pi_w](x) = \sum_{\beta \in \mathcal{R}^+ \cap w\mathcal{R}^+} \beta[k_\beta G_0^{r_\beta w}(r_\beta x) - \frac{2c'}{|\beta|^2}\frac{\pi_w(x)}{(\beta^\vee, x)}]. \quad (16)$$

Now we need the following elementary lemma.

Lemma 3.6 *Let $\alpha \in w\Pi$. Assume that there exists $\beta \in \mathcal{R}_0^+ \cap w\mathcal{R}^+$, such that $\alpha \leq_w \beta$. Then there exists $\gamma \in \Pi \cap w\mathcal{R}^+$, such that $\alpha \leq_w \gamma$.*

Proof of the lemma: Let $\beta = \sum_{\gamma \in \Pi} n_\gamma \gamma$ be the decomposition of β in Π. Since $\beta \in w\mathcal{R}_0^+$, there exists $\gamma \in \Pi \cap w\mathcal{R}_0^+$ such that $n_\gamma > 0$. We see moreover that $\tilde{\gamma} := \sum_{\gamma \in \Pi \cap w\mathcal{R}_0^+} n_\gamma \gamma \in w\mathcal{R}^+$, and that $\beta \leq_w \tilde{\gamma}$, which concludes the proof of the lemma. ∎

Suppose now that there does not exist $\gamma \in \Pi \cap w\mathcal{R}^+$ such that $\alpha_0 \leq_w \gamma$. Then by Lemma 3.6, no other $\beta \in \mathcal{R}^+ \cap w\mathcal{R}^+$ satisfies $\alpha_0 \leq_w \beta$. Thus from equation (16) we get that for all y in the segment between $x_0 - \epsilon \alpha_0^*$ and x_0, $\partial_{\alpha_0^*}[G_0^w - c'\pi_w](y) = 0$, which contradicts the initial hypothesis on x_0. We conclude that there exists $\gamma \in \Pi \cap w\mathcal{R}^+$ such that $\alpha_0 \leq_w \gamma$. Again from (16) we get

$$\partial_{\alpha_0^*}[G_0^w - c'\pi_w](x_0) \geq c_1 c \pi_{r_\gamma w}(r_\gamma x_0) - c_2 c' \frac{\pi_w(x_0)}{(\alpha_0^\vee, x_0)},$$

where c_1 and c_2 are positive constants. But with the first point of Lemma 3.5 we have $\pi_{r_\gamma w}(r_\gamma x_0) = \frac{\pi_w(x_0)}{(\gamma^\vee, x_0)}$. Moreover by our choice of α_0, we have $(\gamma, x_0) \leq |\gamma|_w (\alpha_0, x_0)$. Thus if c' is sufficiently small we get

$$\partial_{\alpha_0^*}[G_0^w - c'\pi_w](x_0) > 0$$

and a contradiction. The induction hypothesis for w follows.

Putting now the third steps together, we get the desired estimate of G_0 away from the walls. With Lemma 3.3, this concludes the proof of the theorem. ∎

The preceding theorem has for us a very important consequence. Let E be the Euler operator. It is defined for f regular, and $x \in \mathfrak{a}$, by $Ef(x) = (x, \nabla f(x))$. The following theorem generalizes the analogue result of [3] in the setting of symmetric spaces. Our proof is in a certain sense more elementary than in [3], because we do not make use of the descent technique of Harish-Chandra.
The first claim of the theorem will be needed in the estimate of the heat semigroup (Proposition 5.2). It will also be used in the study of the asymptotic convergence of the F_0-processes (see [3]). It will allow us in [19] to generalize some results of Anker, Bougerol, and Jeulin [3] for all $k > 0$. The second claim is just a technical result needed in the proof of the estimate of the heat kernel (see Theorem 5.2).

Theorem 3.3 *1. There exists a constant $K > 0$ such that for any $x \in \overline{\mathfrak{a}_+}$,*

$$0 \leq E[\log(e^\rho F_0)](x) \leq K.$$

2. *We have the two following estimates*

$$E[\log(e^{\rho}F_0)](x) = |\mathcal{R}_0^+| + \mathcal{O}(\frac{1}{1+\min_{\alpha\in\mathcal{R}^+}(\alpha,x)}),$$

$$\sum_{\alpha\in\mathcal{R}_0^+} \frac{(\alpha,x)}{\sqrt{1+(\alpha,x)^2}}\partial_\alpha(\log(e^{\rho}F_0))(x) \asymp \frac{1}{1+\min_{\alpha\in\mathcal{R}^+}(\alpha,x)}.$$

Proof of the theorem: With the formula (3) and (9), we get for any $x \in \overline{\mathfrak{a}_+}$,

$$E[\log(e^{\rho}F_0)](x) = \frac{1}{|W|}\sum_{w\in W}[(\rho,x)-(\rho,wx)]\frac{G_0(wx)}{F_0(x)}. \tag{17}$$

1. Formula (17) proves already the first inequality. For the second inequality we show by induction on the length $l(w)$ of $w \in W$ that for all $x \in \overline{\mathfrak{a}_+}$,

$$(\rho,x)-(\rho,wx) \leq K'l(w) \max_{\alpha\in\mathcal{R}_0^+\cap w\mathcal{R}^-} |(\alpha,wx)|, \tag{18}$$

   where $K' = \max_{\alpha\in\mathcal{R}_0^+}(\rho,\alpha^\vee)$ is a constant. Suppose that the induction hypothesis is true for all w of length less or equal to l. Let $v \in W$ be of length $l+1$. Let $\alpha \in \Pi \cap v\mathcal{R}^-$, and let $w = r_\alpha v$. We have $l(w) = l$. Moreover since $\alpha \in \Pi$, r_α maps $\mathcal{R}_0^+ \cap w\mathcal{R}_0^-$ onto $(\mathcal{R}_0^+ \cap v\mathcal{R}_0^-) \setminus \{\alpha\}$. But for all $x \in \mathfrak{a}$,

$$(\rho,x)-(\rho,vx) = (\rho,x)-(\rho,wx)-(\alpha,vx)(\rho,\alpha^\vee).$$

   Thus (18) follows for v by using the induction hypothesis. Now with Theorems 3.1 and 3.2, the first claim is proved.

2. These estimates result also from Formula (17) and the global estimates (Theorems 3.1 and 3.2) of G_0 and F_0. The fact that $|\mathcal{R}_0^+|$ is the limit of $E[\log(e^{\rho}F_0)](x)$ when $(\alpha,x)\to\infty$ for all α can be seen exactly like in [3] by expanding the functions F_λ in series. This finishes the proof of the theorem. ■

3.3 Estimates of the derivatives

In this subsection we estimate the derivatives of the hypergeometric function $G_\lambda(x)$, first in x alone and next jointly in (λ,x).

Proposition 3.2 *Let p be a polynomial of degree N. Then there exists a constant C such that, for any $\lambda \in \mathfrak{h}$ and for any $x \in \mathfrak{a}$,*

$$|p(\frac{\partial}{\partial x})G_\lambda(x)| \leq C(1+|\lambda|)^N F_0(x)e^{\max_w \Re(w\lambda,x)}.$$

Proof of the proposition: According to Proposition 3.1, we know that this estimate holds with no derivative.

Step 1 *: Estimate away from walls*
By induction, Formula (4) allows us to express on $\mathfrak{a}_{\mathrm{reg}}$ derivatives of G_λ in terms of lower order derivatives and to estimate them away from walls. More precisely we obtain this way the desired estimate when x stays at distance $\geq \frac{\epsilon}{1+|\lambda|}$ from walls.
Step 2 *: Estimate on faces*
Assume that x lies in a face $\mathfrak{a}^I$ (of minimal dimension), then (4) becomes (5), which writes also

$$\begin{aligned}\partial_{A_{w,I}(\xi)} G_\lambda(x) &= \sum_{\alpha\in\mathcal{R}^+\smallsetminus\mathcal{R}_I} k_\alpha \frac{(\alpha,\xi)}{1-e^{-(\alpha,x)}}(G_\lambda(r_\alpha x)-G_\lambda(x)) \qquad (19)\\ &+ (\rho+\lambda,\xi)G_\lambda(x),\end{aligned}$$

where

$$A_{w,I}(\xi) = \xi + 2\sum_{\alpha\in\mathcal{R}_I}\frac{k_\alpha}{|\alpha|^2}(\alpha,\xi)\alpha.$$

Notice that the linear map $A_{w,I}:\mathfrak{a}\to\mathfrak{a}$ is one-to-one, since the expression

$$(A_{w,I}(\xi),\xi) = |\xi|^2 + 2\sum_{\alpha\in\mathcal{R}_I}\frac{k_\alpha}{|\alpha|^2}(\alpha,\xi)^2$$

is strictly positive for all nonzero ξ. By induction, (19) yields the following estimate: for every $\epsilon>0$, there exists a constant $C\geq 0$ such that, for all multi-indices κ, for all $\lambda\in\mathfrak{h}$ and for $x\in\mathfrak{a}^I$ such that $\min_{\alpha\in\mathcal{R}^+\smallsetminus\mathcal{R}_I}|(\alpha,x)|\geq\frac{\epsilon}{1+|\lambda|}$,

$$|(\frac{\partial}{\partial x})^\kappa G_\lambda(x)| \leq |\kappa|!C^{|\kappa|}(1+|\lambda|)^{|\kappa|}F_0(x)e^{\max_{w\in W}(w\Re\lambda,x)}. \qquad (20)$$

Step 3 *: Estimate near the faces*
If x is near a face $\mathfrak{a}^I$, we use (20) and the Taylor development of G_λ in the orthogonal projection of x on $\mathfrak{a}^I$. More precisely let $\epsilon>0$ be such that $C\epsilon<1$, where C is the constant appearing in (20). Then there exists a constant $C'>0$ such that, for all multi-indices κ, for all $\lambda\in\mathfrak{h}$ and for $x\in\mathfrak{a}$ at distance $\leq\frac{\epsilon}{1+|\lambda|}$ from $\mathfrak{a}^I$, such that $\min_{\alpha\in\mathcal{R}^+\smallsetminus\mathcal{R}_I}|(\alpha,x)|\geq\frac{\epsilon}{1+|\lambda|}$,

$$|(\frac{\partial}{\partial x})^\kappa G_\lambda(x)| \leq C'(1+|\lambda|)^{|\kappa|}F_0(x)e^{\max_{w\in W}(w\Re\lambda,x)}. \qquad (21)$$

Step 4 *: Conclusion*
Now we first use the step 3 near the origin. We get $\epsilon_0>0$ and $C_0>0$, such that (21) holds (with C_0 in place of C') for $x\in\mathfrak{a}$ at distance $\leq\frac{\epsilon_0}{1+|\lambda|}$ from the origin. Then we use the step 3 near the faces of dimension 1. We get ϵ_1 and C_1 such that (21) holds for $x\in\mathfrak{a}$ at distance $\leq\frac{\epsilon_1}{1+|\lambda|}$ from any face of dimension 1, and at distance $\geq\frac{\epsilon_0}{1+|\lambda|}$ from the origin. And like this we get successively,

for each $d \in \mathbb{N}$, constants $\epsilon_d > 0$ and C_d associated to the faces of dimension d. Eventually we conclude with the first step. ∎

We can now derive the fundamental estimate:

Theorem 3.4 *Let p and q be polynomials of degree M and N. Then there exists a constant C such that, for all $\lambda \in \mathfrak{h}$ and for all $x \in \mathfrak{a}$,*

$$|p(\frac{\partial}{\partial \lambda})q(\frac{\partial}{\partial x})G_\lambda(x)| \leq C(1+|x|)^M(1+|\lambda|)^N F_0(x)e^{\max_w \Re(w\lambda,x)}.$$

Proof of the Theorem: The proof is standard. Theorem 3.4 is deduced from Proposition 3.2 using Cauchy's formula. More precisely, one integrates $G_\lambda(x)$ in the variable λ over n-tori with radii comparable to $\frac{1}{1+|x|}$. ∎

Remark 3.2 This estimate holds true for F_λ too.

4 Hypergeometric Fourier transform and Schwartz spaces

We first recall the definitions of the hypergeometric Fourier transform and of its inverse, according to Cherednik [7]. Let μ be the measure on $\mathfrak{a}$ given by

$$d\mu(x) = \underbrace{\prod_{\alpha\in\mathcal{R}^+} |2\sinh(\frac{\alpha}{2},x)|^{2k_\alpha}}_{:=\delta(x)} dx.$$

The hypergeometric Fourier transform $\mathcal{H}$ is defined for nice functions f on $\mathfrak{a}$ by

$$\mathcal{H}(f)(\lambda) = \int_{\mathfrak{a}} f(x)G_\lambda(-x)d\mu(x),\ \forall\lambda\in\mathfrak{h}. \tag{22}$$

Let ν be the asymmetric Plancherel measure on $i\mathfrak{a}$ defined by

$$d\nu(\lambda) = c\prod_{\alpha\in\mathcal{R}^+}\frac{\Gamma((\lambda,\alpha^\vee)+k_\alpha+\frac{1}{2}k_{\frac{\alpha}{2}})\Gamma(-(\lambda,\alpha^\vee)+k_\alpha+\frac{1}{2}k_{\frac{\alpha}{2}}+1)}{\Gamma((\lambda,\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}})\Gamma(-(\lambda,\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}}+1)}d\lambda,$$

where c is a normalizing constant. The inverse transform $\mathcal{I}$ is given for nice functions h by

$$\mathcal{I}(h)(x) = \int_{i\mathfrak{a}} h(\lambda)G_\lambda(x)d\nu(\lambda),\ \forall x\in\mathfrak{a}. \tag{23}$$

In the case $k=0$, $\mathcal{H}$ and $\mathcal{I}$ reduce to the classical Euclidean Fourier transform

$$\mathcal{F}(f)(\lambda) = \int_{\mathfrak{a}} f(x)e^{-(\lambda,x)}dx$$

and its inverse
$$\mathcal{F}^{-1}(h)(x) = (2\pi)^{-n} \int_{i\mathfrak{a}} h(\lambda) e^{(\lambda,x)} d\lambda.$$
We shall consider the following function spaces. The classical Schwartz space on $i\mathfrak{a}$ is denoted by $\mathcal{S}(i\mathfrak{a})$. Its topology is defined by the semi-norms

$$\tau_{p,N}(h) = \sup_{\lambda \in i\mathfrak{a}} (1+|\lambda|)^N |p(\frac{\partial}{\partial \lambda}) h(\lambda)|,$$

where p is any polynomial and $N \in \mathbb{N}$. As usual $C_c^\infty(\mathfrak{a})$ denotes the space of C^∞ functions on $\mathfrak{a}$ with compact support and $C_\Gamma^\infty(\mathfrak{a})$ the subspace of functions with support in a given compact subset Γ. Let us denote by $\mathcal{C}(\mathfrak{a})$ the space of C^∞ functions on $\mathfrak{a}$, such that for all polynomials p and all $N \in \mathbb{N}$,

$$\sup_{x \in \mathfrak{a}} (1+|x|)^N F_0(x)^{-1} |p(\frac{\partial}{\partial x}) f(x)| < +\infty,$$

It is the Schwartz space on $\mathfrak{a}$ associated to the measure μ. Its topology is defined by the semi-norms

$$\sigma_{p,N}(f) = \sup_{x \in \mathfrak{a}} (1+|x|)^N F_0(x)^{-1} |p(\frac{\partial}{\partial x}) f(x)|.$$

Notice that according to Proposition 3.1, we may replace $F_0(x)$ by $e^{-(\rho,x^+)}$ in the definition of $\mathcal{C}(\mathfrak{a})$ and its topology. Let us recall that x^+ is the only point in the orbit $W \cdot x$ which lies in $\overline{\mathfrak{a}_+}$.

Lemma 4.1 *1. $\mathcal{C}(\mathfrak{a})$ is a Fréchet space.*

2. $C_c^\infty(\mathfrak{a})$ is a dense subspace of $\mathcal{C}(\mathfrak{a})$.

Proof of the lemma: These facts are standard. The second one is proved for example in [8], more precisely in Appendix A by M. Tinfou. ■

Eventually, the Paley-Wiener space $PW(\mathfrak{h})$ consists of all entire functions h on $\mathfrak{h}$ which satisfy the following growth condition:

$$\exists R \geq 0, \ \forall N \in \mathbb{N}, \ \sup_{\lambda \in \mathfrak{h}} (1+|\lambda|)^N e^{-R|\Re\lambda|} h(\lambda) < \infty.$$

Given a W-invariant convex compact subset Γ in $\mathfrak{a}$, $PW_\Gamma(\mathfrak{h})$ denotes the subspace of $PW(\mathfrak{h})$ defined by the specific condition

$$\forall N \in \mathbb{N}, \ \sup_{\lambda \in \mathfrak{h}} (1+|\lambda|)^N e^{-\gamma(-\Re\lambda)} h(\lambda) < \infty.$$

Here $\gamma(\lambda) = \sup_{x \in \Gamma} (\lambda, x)$ is the gauge associated to the polar of Γ.
The mapping properties of the hypergeometric Fourier transform were investigated by Opdam [14] and revisited by Cherednik [6]. Here are two main results

(i) Paley-Wiener theorem: $\mathcal{H}$ and $\mathcal{I}$ are (up to positive constants) inverse isomorphisms between $C_c^\infty(\mathfrak{a})$ and $PW(\mathfrak{h})$.

(ii) Plancherel type formula:

$$\int_{\mathfrak{a}} f(x)g(-x)d\mu(x) = \text{const} \cdot \int_{i\mathfrak{a}} \mathcal{H}f(\lambda)\mathcal{H}g(\lambda)d\nu(\lambda).$$

Opdam [14] proved eventually a more precise Paley-Wiener theorem: $\mathcal{H}$ and $\mathcal{I}$ map $C_\Gamma^\infty(\mathfrak{a})$ and $PW_\Gamma(\mathfrak{h})$ into each other (and hence are inverse maps, up to a positive constant), where Γ is the convex hull of any W-orbit $W \cdot x$ in $\mathfrak{a}$. The proof works as well for the polar sets

$$\Gamma = \{x \in \mathfrak{a} \mid (\Lambda^+, x^+) \leq 1\}$$

where Λ is any regular element in $\mathfrak{a}$. We shall need this version of the Paley-Wiener theorem with positive multiples of ρ.
We are now able to resume Anker's approach [2] in order to analyze the hypergeometric Fourier transform in the Schwartz class. The following type of result was already obtained by Delorme [8], following Harish-Chandra's strategy. On one hand, Delorme considers only W-invariant functions but, on the other hand, he deals with the more difficult case where $k < 0$.

Theorem 4.1 *The hypergeometric Fourier transform $\mathcal{H}$ and its inverse $\mathcal{I}$ are topological isomorphisms between $\mathcal{C}(\mathfrak{a})$ and $\mathcal{S}(i\mathfrak{a})$.*

Sketch of the proof: The proof is divided in two parts which correspond to the following two lemmas. The first one is elementary.

Lemma 4.2 *The hypergeometric Fourier transform $\mathcal{H}$ maps $\mathcal{C}(\mathfrak{a})$ continuously into $\mathcal{S}(i\mathfrak{a})$.*

Lemma 4.3 *The inverse transform $\mathcal{I} : PW(i\mathfrak{a}) \to C_c^\infty(\mathfrak{a})$ is continuous for the topology inherited from $\mathcal{S}(i\mathfrak{a})$ and $\mathcal{C}(\mathfrak{a})$ respectively.*

Proof of the lemma: Let $h \in PW$ and $f = \mathcal{I}(h)$. Given a semi-norm $\sigma = \sigma_{p,N}$ on $\mathcal{C}(\mathfrak{a})$, we must find a semi-norm τ on $\mathcal{S}(i\mathfrak{a})$ such that

$$\sigma_{p,N}(f) \leq \tau(h).$$

We denote by g the image of h by the inverse Euclidean Fourier transform $\mathcal{F}^{-1}$. According to the Paley-Wiener theorems for the hypergeometric and the Euclidean Fourier transforms, we have the following support conservation property: supp(f) is contained in $\Gamma_r = \{x \in \mathfrak{a} \mid (\rho, x^+) \leq r\}$ if and only if supp$(g) \subset \Gamma_r$. Let $\omega_j \in C^\infty(\mathfrak{a})$ such that $\omega_j = 0$ inside Γ_{j-1}, $\omega_j = 1$ outside Γ_j, and ω_j is uniformly bounded in $j \in \mathbb{N}^*$, as well as each derivative. Set $g_j = \omega_j g$, $h_j = \mathcal{F}(g_j)$ and $f_j = \mathcal{I}(h_j)$. Here is a crucial observation: we have $g_j = g$ outside Γ_j, hence $f_j = f$ outside Γ_j, by the above support property. Let us estimate $f = f_j$ on $\Gamma_{j+1} \setminus \Gamma_j$. First of all, using Proposition 3.2, there exist $N' \in \mathbb{N}$ and $C > 0$ such that

$$\sup_{x\in\Gamma_{j+1}\setminus\Gamma_j} (1+|x|)^N F_0(x)^{-1} |p(\frac{\partial}{\partial x}) f_j(x)| < C j^N \tau_{1,N'}(h_j).$$

Next, by the Euclidean Fourier analysis

$$\tau_{1,N'}(h_j) \leq C \sum_{l=0}^{N'} \sup_{x\in\mathfrak{a}} (|x|+1)^{n+1} |\nabla^l g_j(x)|.$$

Observe that g_j and its derivatives vanish in Γ_{j-1}. Hence

$$j^N \tau_{1,N'}(h_j) \leq C \sum_{l=0}^{N'} \sup_{x\in\mathfrak{a}\setminus\Gamma_{j-1}} (|x|+1)^{N+n+1} |\nabla^l g(x)|.$$

Again, by Euclidean Fourier analysis,

$$\sum_{l=0}^{N'} \sup_{x\in\mathfrak{a}} (|x|+1)^{N+n+1} |\nabla^l g(x)| \leq C \tau_{N+n+1,N''}(h).$$

In summary, there exist $N'' \in \mathbb{N}$ and $C > 0$ such that, for every $j \in \mathbb{N}^*$,

$$\sup_{x\in\Gamma_{j+1}\setminus\Gamma_j} (1+|x|)^N F_0(x)^{-1} |p(\frac{\partial}{\partial x}) f(x)| \leq C \tau_{N+n+1,N''}(h).$$

The remaining estimate of f in Γ_1 is elementary. ■

In the W-invariant setting, the hypergeometric Fourier transform and its inverse write

$$\mathcal{H}(f)(\lambda) = \int_{\mathfrak{a}} f(x) F_\lambda(-x) d\mu(x)$$

and

$$\mathcal{I}(h)(\lambda) = \int_{i\mathfrak{a}} h(\lambda) F_\lambda(x) d\nu'(\lambda)$$

where

$$\begin{aligned} d\nu'(\lambda) &= \text{const} \cdot \prod_{\alpha\in\mathcal{R}+} \frac{\Gamma((\lambda,\alpha^\vee)+k_\alpha+\frac12 k_{\frac{\alpha}{2}})\Gamma(-(\lambda,\alpha^\vee)+k_\alpha+\frac12 k_{\frac{\alpha}{2}})}{\Gamma((\lambda,\alpha^\vee)+\frac12 k_{\frac{\alpha}{2}})\Gamma(-(\lambda,\alpha^\vee)+\frac12 k_{\frac{\alpha}{2}})} d\lambda \\ &= \text{const} \cdot \mathbf{c}(\lambda)^{-1} \mathbf{c}(-\lambda)^{-1}\ d\lambda \end{aligned}$$

is the symmetric Plancherel measure or Harish-Chandra measure (see [7]). We denote by $\mathcal{C}(\mathfrak{a})^W$ and $\mathcal{S}(i\mathfrak{a})^W$ the spaces of W-invariant functions of $\mathcal{C}(\mathfrak{a})$ and $\mathcal{S}(i\mathfrak{a})$ respectively, which we identify also with their restriction to $\overline{\mathfrak{a}_+}$. From Theorem 4.1 we get

Corollary 4.1 *These transforms are topological isomorphisms between $\mathcal{C}(\mathfrak{a})^W$ and $\mathcal{S}(i\mathfrak{a})^W$.*

We recover this way the main result of [8] in the easy case $k > 0$.

5 The heat kernel

5.1 Solution to the Cauchy problem

In this section we solve the heat equation (with Cauchy data) for the Heckman-Opdam Laplacian. We follow essentially the presentation of Rösler [17] section 4, and refer to this article for some proofs, which are identical in our setting. We denote by $\mathcal{D}$ the modified Laplacian defined by

$$\mathcal{D} = \frac{1}{2}(\mathcal{L} - |\rho|^2).$$

The heat operator H is defined by

$$H = \partial_t - \mathcal{D}$$

on $C^{2,1}(\mathfrak{a} \times \mathbb{R})$. We consider the standard Cauchy problem: Given a continuous bounded function f on $\mathfrak{a}$, find $u \in C^{2,1}(\mathfrak{a} \times (0, +\infty)) \cap C^0(\mathfrak{a} \times [0, +\infty))$, such that

$$\begin{cases} Hu = 0 & \text{on } \mathfrak{a} \times (0, +\infty) \\ u(\cdot, 0) = f. \end{cases} \tag{24}$$

Definition 5.1 *The heat kernel $p_t(x, y)$ is defined for $x, y \in \mathfrak{a}$ and $t > 0$ by*

$$p_t(x, y) = \int_{i\mathfrak{a}} e^{-\frac{t}{2}(|\lambda|^2+|\rho|^2)} G_\lambda(x) G_\lambda(-y) d\nu(\lambda). \tag{25}$$

The heat semigroup $(P_t, t \geq 0)$ is defined for $f \in \mathcal{C}(\mathfrak{a})$ and $t \geq 0$ by

$$P_t f(x) := \begin{cases} \int_{\mathfrak{a}} p_t(x, y) f(y) d\mu(y) & \text{if } t > 0 \\ f(x) & \text{if } t = 0. \end{cases}$$

Using the hypergeometric Fourier transform and its inverse, we can express the heat semigroup as follows

$$P_t f = \mathcal{I}(\lambda \mapsto e^{-\frac{t}{2}(|\lambda|^2+|\rho|^2)} \mathcal{H}(f)(\lambda))$$

and deduce its basic properties which are summarized in the following theorem (the analogue of Theorem 4.7 in [17]).

Theorem 5.1 *1. $(P_t, t \geq 0)$ is a strongly continuous semigroup on $\mathcal{C}(\mathfrak{a})$.*

2. Let $f \in \mathcal{C}(\mathfrak{a})$. Then $u(x, t) = P_t f(x)$ solves the Cauchy problem (24).

As in the Dunkl setting, we show next that $(P_t, t \geq 0)$ can be extended to a strongly continuous semigroup on $C_0(\mathfrak{a})$ (the space of continuous functions $f : \mathfrak{a} \to \mathbb{C}$ which vanish at infinity, equipped with the norm $|f|_\infty = \sup_{x \in \mathfrak{a}} |f(x)|$). Consider $\mathcal{D}$ as a densely defined linear operator on $C_0(\mathfrak{a})$ with domain $\mathcal{C}(\mathfrak{a})$.

Proposition 5.1 *1. The operator $(\mathcal{D}, \mathcal{C}(\mathfrak{a}))$ has a closure, which generates a Feller semigroup $(T(t), t \geq 0)$ on $C_0(\mathfrak{a})$.*

2. *$T(t)$ coincides with P_t on $\mathcal{C}(\mathfrak{a})$.*

Proof of the Proposition:

1. In order to apply the Hille-Yosida Theorem (see [9] Theorem 2.2 p.165) we need to check the following two properties:

   (a) Let $f \in \mathcal{C}(\mathfrak{a})$. Assume that x_0 is a global maximum of f. Then $\mathcal{D}f(x_0) \leq 0$ (this is the positive maximum principle).

   (b) $(\mu I - \mathcal{D})(\mathcal{C}(\mathfrak{a}))$ is dense in $C_0(\mathfrak{a})$ for some $\mu > 0$.

   (a) follows from the explicit expression (1) of $\mathcal{L}$. For (b) we prove with Theorem 4.1 that $(\mu I - \mathcal{D})$ maps $\mathcal{C}(\mathfrak{a})$ onto itself for every $\mu > 0$. In fact if $f \in \mathcal{C}(\mathfrak{a})$, then

$$\mathcal{H}((\mu I - \mathcal{D})f)(\lambda) = (\mu + \frac{|\rho|^2 + |\lambda|^2}{2})\mathcal{H}(f)(\lambda), \ \lambda \in i\mathfrak{a}.$$

2. The equality $T(t)f = P_t f$ results from the uniqueness of solution to (24) within the class of all differentiable functions on $[0, \infty)$ with values in $C_0(\mathfrak{a})$ (see [17]). ■

Corollary 5.1 *The heat kernel $p_t(x, y)$ is positive on $\mathfrak{a} \times \mathfrak{a} \times (0, \infty)$, symmetric in (x, y), and satisfies the following properties:*

1. *For all $x, y \in \mathfrak{a}$, for all $t > 0$ and all $w \in W$, $p_t(wx, wy) = p_t(x, y)$.*
2. *For all $t > 0$ and $x \in \mathfrak{a}$, $p_t(x, \cdot) \in \mathcal{C}(\mathfrak{a})$.*
3. *Let $f \in C_b(\mathfrak{a})$. Then*

$$u(x,t) = P_t f(x) = \begin{cases} \int_{\mathfrak{a}} p_t(x, y) f(y) d\mu(y) & \text{if } t > 0 \\ f(x) & \text{if } t = 0 \end{cases}$$

   is still a solution to the Cauchy problem (24).

4. *For all $t > 0$ and all $x \in \mathfrak{a}$, $\int_{\mathfrak{a}} p_t(x, y) d\mu(y) = 1$.*

Proof of the corollary: The positivity property results from the last proposition, which implies that $P_t f \geq 0$ for any $f \in \mathcal{C}(\mathfrak{a})$ with $f \geq 0$. Thus (see [17]) $p_t(x, y) \geq 0$ for all $t > 0$ and $x, y \in \mathfrak{a}$, by continuity of $p_t(x, \cdot)$. The invariance of p_t under the Weyl group results from the invariance of $\mathcal{D}$ when $\mathcal{R}^+$ is replaced by $w\mathcal{R}^+$, for any $w \in W$. The symmetry of p_t results in the same way from its invariance by $-Id$, and from Formula (25). The second and third assumptions are classical and result from basic properties of the G_λ (see [17]). The last assumption results from the point 3 and the fact that $T(t)1 = 1$ (because $\mathcal{D}$ is conservative, see [9] p.166). ■

The W-invariant heat kernel p_t^W is defined for all $x, y \in \mathfrak{a}$ and $t > 0$ by

$$\begin{aligned} p_t^W(x,y) &= \sum_{w\in W} p_t(x,wy) = \frac{1}{|W|}\sum_{w,w'\in W} p_t(wx,w'y) \\ &= \int_{i\mathfrak{a}} e^{-\frac{t}{2}(|\lambda|^2+|\rho|^2)} F_\lambda(x)F_\lambda(-y)d\nu'(\lambda). \end{aligned}$$

The W-invariant semigroup $(P_t^W, t \geq 0)$ is defined for $f \in \mathcal{C}(\overline{\mathfrak{a}_+})$, $x \in \overline{\mathfrak{a}_+}$ and $t \geq 0$, by

$$P_t^W f(x) = \int_{\mathfrak{a}_+} p_t^W(x,y)f(y)d\mu(y), \text{ if } t > 0,$$

and $P_0^W f(x) = f(x)$. We have naturally the analogue of Theorem 5.1. The generator of $(P_t^W, t \geq 0)$ is equal on $\mathcal{C}(\mathfrak{a})^W$ to the differential part D of $\mathcal{D}$. The analogue of Proposition 5.1 for D, is a consequence of Corollary 4.1 and the following lemma. The second claim of this lemma will be used in [19].

Lemma 5.1 *The space $\mathcal{C}(\mathfrak{a})^W$ is dense in $C_0(\overline{\mathfrak{a}_+})$. Moreover if $f \in C_c^\infty(\overline{\mathfrak{a}_+})$, there exists a sequence $(u_j)_j \in \mathcal{C}(\overline{\mathfrak{a}_+})^W$ which converges uniformly to f, and which satisfies: there exists a positive constant $C > 0$, independent of j, such that $|\nabla u_j(x)| \leq C$ for all $x \in \overline{\mathfrak{a}_+}$, and if $d(x, \partial\mathfrak{a}_+) > \frac{1}{j}$, then $|\Delta u_j(x)| \leq C$, whereas if $d(x, \partial\mathfrak{a}_+) \leq \frac{1}{j}$, then $|\frac{\Delta u_j(x)}{j}| \leq C$.*

Proof of the lemma: The density result is a consequence of the Stone-Weierstrass theorem. However here we need more information, so we need the usual technique of regularization by convolution with an approximate of unity. Let $f \in C_c^\infty(\overline{\mathfrak{a}_+})$. We can extend it to $\mathfrak{a}$ by W-symmetry, and we get a function $\tilde{f}$ which is symmetric, and Lipschitz. Let u be an approximate of unity, which is a W-symmetric C^∞ function, with compact support in the unit ball, and with integral equal to one. Then we consider the sequence of functions $(u_j)_j$ defined by $u_j(x) := \int_{\mathfrak{a}} \tilde{f}(x-y)j^n u(jy)dy$ for $x \in \mathfrak{a}$. It is classical to see that u_j is C^∞ and converges uniformly to $\tilde{f}$. It is also immediate that u_j is W-symmetric. To see that it has the required properties, observe that $\tilde{f}$ is derivable (in the sense of distributions) with a bounded derivative near the walls, and it is C^∞ away from the walls. ■

We set $h_t(x) = p_t(0,x) = \frac{1}{|W|}p_t^W(0,x)$ for $x \in \overline{\mathfrak{a}_+}$, and $t > 0$. We have the formula:

$$h_t(x) = \int_{i\mathfrak{a}} e^{-\frac{t}{2}(|\lambda|^2+|\rho|^2)} F_\lambda(x)d\nu'(\lambda). \tag{26}$$

We will now prove that the heat kernel is in fact strictly positive. As usually we will prove this fact by using a strong minimum principle. The result may be found in [15], but stated in a slightly different way. Thus we include a proof.

Lemma 5.2 (Strong minimum principle) *Let $t_0 \in \mathbb{R}$. Let $u \in C^{2,1}(\mathfrak{a} \times (t_0, +\infty)) \cap C(\mathfrak{a} \times [t_0, +\infty))$. Assume that $Hu \geq 0$ on $\mathfrak{a} \times (t_0, +\infty)$, $u \geq 0$ on $\mathfrak{a} \times [t_0, +\infty)$, and $u(0,t) > 0$, for all $t \geq t_0$. Then $u > 0$ on $\mathfrak{a} \times (t_0, +\infty)$.*

Proof of the lemma: Consider the ellipsoid

$$E \ : \ |x|^2 + \gamma(t-t_0)^2 < \delta.$$

Assume that $u > 0$ on E, and that $u(x_*, t_*) = 0$ for some $(x_*, t_*) \in \partial E$, with $t* > t_0$. By hypothesis (x_*, t_*) can not be the north pole. Moreover by reducing E if necessary, we can always suppose that it is the only point in $\overline{E} \cap \{t > t_0\}$ where u vanishes. We shall perturb u in a small ball

$$B \ : \ |x - x_*|^2 + (t - t_*)^2 < \epsilon^2,$$

with $0 < \epsilon < \min(\frac{1}{2}|x_*|, \frac{1}{2}(t^* - t_0)^2)$. Consider the auxiliary function

$$\omega(x,t) = e^{-r\delta} - e^{-r\{|x|^2 + \gamma(t-t_0)^2\}}.$$

Let us compute and estimate

$$\begin{aligned} H\omega(x,t) &= 2r\Big\{2r|x|^2 - 1 + \gamma(t-t_0) - \sum_{\alpha \in \mathcal{R}^+} k_\alpha(\alpha, x)\coth(\frac{\alpha}{2}, x)\Big\} \\ &\times \ e^{-r\{|x|^2+\gamma(t-t_0)^2\}}. \end{aligned}$$

This expression can be made strictly positive on $\overline{B}$, by choosing $r > 0$ sufficiently large. The function $v = u + \epsilon'\omega$

- is strictly positive on $\overline{B} \setminus \overline{E}$, since $\omega > 0$ outside of $\overline{E}$,
- is equal to u on $\overline{B} \cap \partial E$, since ω vanishes on ∂E,
- can be made strictly positive on $\partial B \cap \overline{E}$ by choosing $\epsilon' > 0$ sufficiently small.

Thus the minimum $v_* \leq 0$ of v on $\overline{B}$ is achieved at an inner point. There $\partial_t v = 0$, $\nabla v = 0$, and $\Delta v \geq 0$. Hence $Hv \leq 0$. But on the other side $Hv = Hu + \epsilon' H\omega > 0$, and we have a contradiction. ■

We can deduce from this lemma the

Corollary 5.2 *The heat kernel $p_t(x,y)$ is strictly positive on $\mathfrak{a} \times \mathfrak{a} \times (0, +\infty)$.*

Proof of the corollary: First we apply the preceding lemma for the function $u(x,t) = h_t(x)$. We have simply to prove that $h_t(0)$ is strictly positive for all $t > 0$. This comes from Formula (26). Moreover since the preceding lemma may be applied for any $t_0 > 0$, we get that $p_t(x, 0) > 0$ for any $t > 0$ and $x \in \mathfrak{a}$. Suppose now that $p_t(x,y) = 0$ for some $x, y \in \mathfrak{a} - \{0\}$ and $t > 0$. We have

$$p_t(x,y) = \int_{\mathfrak{a}} p_{\frac{t}{2}}(x,z) p_{\frac{t}{2}}(z,y) d\mu(z).$$

But as p is positive and continuous, this implies that $p_{\frac{t}{2}}(x,0)p_{\frac{t}{2}}(0,y) = 0$, and we get a contradiction. ■

Remark 5.1 Since the space $\mathcal{C}(\mathfrak{a})$ is dense in all the $L^p(\mathfrak{a},\mu)$ spaces, for $p \in [1,\infty)$, the Hille-Yosida theorem (cf [9]) assures that $\mathcal{D}$ is closable on $L^p(\mathfrak{a},\mu)$ and generates a heat semigroup $(T^{(p)}(t), t \geq 0)$, which is strongly continuous. Moreover, still by an argument of uniqueness in the Cauchy problem, we see that $T^{(p)}$ coincides with the preceding operator P on $\mathcal{C}(\mathfrak{a})$. And by continuity we see that $T^{(p)}$ is just the natural extension of P on $L^p(\mathfrak{a},\mu)$. It is equal for $f \in L^p(\mathfrak{a},\mu)$, $x \in \mathfrak{a}$, and $t > 0$, to

$$T^{(p)}(t)f(x) = P_t f(x) = \int_{\mathfrak{a}} p_t(x,y)f(y)d\mu(y).$$

Obviously the same discussion apply in the radial situation (with D and P^W in place of $\mathcal{D}$ and P respectively).

5.2 Estimates and asymptotic of the heat kernel

In this subsection we establish a sharp global estimate of h_t (Theorem 5.2) and an asymptotic of $p_T(x,\sqrt{T}y)$ when $T \to \infty$ (Proposition 5.3). Let $\gamma := \sum_{\alpha\in\mathcal{R}_+} k_\alpha$, and like usually for $x \in \mathfrak{a}$, we denote by x^+ its symmetric in $\overline{\mathfrak{a}_+}$. A problem in order to get global estimate of p_t is that it is not a convolution operator. Thus $p_t(\cdot,\cdot)$ can not be simply expressed in terms of the function $h_t(\cdot)$. Therefore the next Theorem is only a partial result. A better one could be obtain if we had a global estimate of the Dunkl kernel.

Theorem 5.2 *The following global estimate holds, for all $t > 0$ and $x \in \mathfrak{a}$:*

$$\begin{aligned} h_t(x) &\asymp t^{-\gamma-\frac{n}{2}} \left\{ \prod_{\alpha\in\mathcal{R}_0^+} (1+|(\alpha,x)|)(1+t+|(\alpha,x)|)^{k_\alpha+k_{2\alpha}-1} \right\} \\ &\times e^{-|\rho|^2\frac{t}{2}-(\rho,x^+)-\frac{|x|^2}{2t}}. \end{aligned}$$

Proof of the theorem: Thanks to Theorem 3.3, and the known expression of the heat kernel associated to the Dunkl Laplacian [17], we can use exactly the same proof as in [5]. In this proof it was made use of the heat kernel in balls of radius $R > 0$ with boundary conditions. This may be avoided by using weak parabolic minimum (or maximum) principles for unbounded domains, which hold also because the heat kernel vanishes at infinity. ∎

Our next result gives an equivalent of $p_T(x,\sqrt{T}y)$ when T tends to ∞. This result will be needed in [19] for the proof of the convergence of the normalized F_0-process. However since the proof is easier in the W-invariant case, we begin by the analogue result for $p_T^W(x,\sqrt{T}y)$. Then we will simply explain what has to be modified in the non invariant setting.

Proposition 5.2 *There exists a constant $K > 0$, such that for any $x \in \overline{\mathfrak{a}_+}$ and any $y \in \mathfrak{a}_+$,*

$$p_T^W(x,\sqrt{T}y) \sim K e^{-\frac{|y|^2}{2}} T^{-\frac{n}{2}-|\mathcal{R}_0^+|} e^{-\frac{|\rho|^2}{2}T} F_0(-x)F_0(\sqrt{T}y),$$

when $T \to +\infty$.

Proof of the proposition: We resume the "analysis away from walls" carried out in [4]. It consists in expanding F_λ in the heat kernel expression

$$p_T^W(x, \sqrt{T}y) = \int_{i\mathfrak{a}} e^{-\frac{T}{2}(|\lambda|^2+|\rho|^2)} F_\lambda(-x) F_\lambda(\sqrt{T}y) d\nu'(\lambda) \tag{27}$$

using the Harish-Chandra series [11]

$$F_\lambda(y) = \sum_{w\in W} \mathbf{c}(w\lambda) e^{(w\lambda-\rho,y)} \sum_{q\in Q^+} \Gamma_q(w\lambda) e^{-(q,y)}.$$

Recall that this expression holds for $y \in \mathfrak{a}_+$. Now we replace $F_\lambda(\sqrt{T}y)$ by its development in series in the integral (27). The properties of the coefficients q_χ allow us to invert the integral term and the series (see [4] for more details). Therefore we get

$$p_T^W(x, \sqrt{T}y) = \sum_{q\in Q^+} E_q(x,y) e^{-\frac{|\rho|^2}{2}T-(\rho+q,\sqrt{T}y)} \tag{28}$$

where (using the W-invariance of ν' in λ), for $x, y \in \mathfrak{a}$,

$$E_q(x,y) = K \int_{i\mathfrak{a}} e^{-\frac{T}{2}|\lambda|^2+(\lambda,\sqrt{T}y)} F_\lambda(-x) c(\lambda) \Gamma_q(\lambda) d\nu'(\lambda).$$

Here K is a constant whose value may change in the sequel. We denote by $\mathbf{b}'$ the function defined by

$$\mathbf{b}'(\lambda) \frac{\mathbf{c}(\lambda)}{\pi(\lambda)} d\nu'(\lambda) = d\lambda.$$

It is holomorphic in zero. Observe now that

$$\pi(\frac{1}{T}\frac{\partial}{\partial_\lambda}) e^{-\frac{T}{2}|\lambda|^2} = \pi(-\lambda) e^{-\frac{T}{2}|\lambda|^2}.$$

This formula comes from the fact that there are no skew symmetric polynomial of strictly lower degree than $|\mathcal{R}_0^+|$. Thus the function E_0 may be rewritten into

$$E_0(x,y) = K \int_{i\mathfrak{a}} e^{-\frac{T}{2}|\lambda|^2} \pi(\frac{1}{T}\frac{\partial}{\partial_\lambda}) \{ e^{(\lambda,\sqrt{T}y)} F_\lambda(-x) b'(\lambda)^{-1} \} d\lambda.$$

Then we make the change of variables $v = \frac{y+\lambda}{\sqrt{T}}$, and we get

$$\begin{aligned} E_0(x,y) e^{-\frac{|\rho|^2}{2}T-(\rho,\sqrt{T}y)} &\sim K F_0(-x) e^{-\frac{T}{2}|\rho|^2-(\rho,\sqrt{T}y)-\frac{|y|^2}{2}} T^{-\frac{D}{2}} \pi(\sqrt{T}y) \\ &\times \int_{i\mathfrak{a}} e^{\frac{1}{2}|v|^2} \frac{F_{\frac{v-y}{\sqrt{T}}}(-x)}{F_0(-x)} b'^{-1}(\frac{v-y}{\sqrt{T}}) dv. \end{aligned}$$

The preceding integral has a finite limit, independent of x and y, when T tends to infinity. Thus using the known asymptotic of F_0 (Theorem 3.1), we conclude that the first term of the series in (28) has the desired asymptotic. A similar study would show that the leading terms are negligible. This concludes the proof of the proposition. ∎

Proposition 5.3 *There exists a constant $K > 0$, such that for any $x \in \mathfrak{a}$, and any $y \in \mathfrak{a}_{reg}$, if $wy \in \mathfrak{a}_+$, then*

$$p_T(x, \sqrt{T}y) \sim Ke^{-\frac{|y|^2}{2}} T^{-\frac{n}{2}-|\mathcal{R}_0^+|} e^{-\frac{|\rho|^2}{2}T} G_0(wx) F_0(\sqrt{T}y),$$

when $T \to +\infty$.

Proof of the proposition: The proof is analogue as for the preceding proposition. First we have $p_T(x, \sqrt{T}y) = p_T(wx, w\sqrt{T}y)$. Then in the integral expression of $p_T(wx, w\sqrt{T}y)$, we replace $G_\lambda(-w\sqrt{T}y)$ by its development in series. Since $-wy \in \mathfrak{a}_-$, we already know the dominant coefficients of the development. Indeed they were computed by Opdam in [14]: they are all null except one which is equal up to a constant to $\pi(\lambda)$. But $\pi(\lambda)d\nu(\lambda)$ behaves like $d\nu'(\lambda)$ in zero, i.e. like $|\pi(\lambda)|^2$. Thus we can follow the rest of the proof of the preceding proposition, and we get the result. ■

5.3 The Poisson equation for $\mathcal{D}$

Our sharp estimates of Theorem 3.4 allows us to prove the

Proposition 5.4 *Let $f \in L^1(\mathfrak{a}, \mu)$. Then the function $Gf : x \mapsto \int_0^\infty P_t f(x) dt$ is finite μ-a.e. If moreover $\mathcal{F}(f) \in L^1(i\mathfrak{a}, \nu)$, then Gf is bounded, belongs to $C^2(\mathfrak{a})$, and satisfies the Poisson equation $\mathcal{D}Gf = -f$.*

Proof of the proposition: Let $f \in L^1(\mathfrak{a}, \mu)$. For all x, and all $\epsilon > 0$, we have

$$\begin{aligned} |Gf(x)| &= |\int_0^\infty e^{-\frac{t}{2}|\rho|^2} \int_{\mathfrak{a}} \int_{i\mathfrak{a}} e^{-\frac{t}{2}|\lambda|^2} G(\lambda, x) G(-\lambda, y) d\nu(\lambda) f(y) d\mu(y) dt| \\ &\leq |\int_0^1 P_t f(x) dt| + C|f|_1 \int_1^\infty e^{-\frac{t}{2}|\rho|^2} |\int_{i\mathfrak{a}} e^{-\frac{t}{2}|\lambda|^2} d\nu(\lambda)| dt, \end{aligned}$$

where C is a constant. But since for any $t \geq 0$, P_t is a contraction on L^1, we have therefore $|P_t f|_1 \leq |f|_1$. Thus $|\int_0^1 P_t f dt|_1 \leq |f|_1 < \infty$. And then μ-a.e., $|\int_0^1 P_t f dt| < \infty$. Finally we get that μ a.e. $Gf < \infty$. This proves the first claim of the proposition. Now let $f \in L^1(\mathfrak{a}, \mu)$, be such that $\mathcal{F}(f) \in L^1(i\mathfrak{a}, \nu)$. Then we have

$$\begin{aligned} |Gf(x)| &\leq \int_{i\mathfrak{a}} \mathcal{F}(f)(\lambda) \int_0^\infty e^{-\frac{t}{2}|(|\lambda|^2+\rho|^2)} dt d\nu(\lambda) \\ &\leq 2\int_{i\mathfrak{a}} \frac{\mathcal{F}(f)(\lambda)}{|\lambda|^2 + |\rho|^2} d\nu(\lambda). \end{aligned}$$

This shows that Gf is bounded. Moreover using a theorem of derivation under the integral, and our precise estimate of the derivatives of the functions G_λ, we see that $Gf \in C^2(\mathfrak{a})$ and satisfies $\mathcal{D}Gf = -f$. This finishes the proof of the proposition. ■

6 Appendix : computation of the Heckman-Opdam laplacian

First we give another expression of the Cherednik operator:

$$\begin{aligned}
T_\xi f(x) &= \partial_\xi f(x) + \sum_{\alpha\in\mathcal{R}^+} \frac{k_\alpha}{2}(\alpha,\xi)\coth\frac{(\alpha,x)}{2}\{f(x)-f(r_\alpha.x)\} \\
&- \sum_{\alpha\in\mathcal{R}^+}\frac{k_\alpha}{2}(\alpha,\xi)f(r_\alpha.x).
\end{aligned}$$

Now we compute

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{R}^+}\frac{k_\alpha}{2}\coth\frac{(\alpha,x)}{2}\{T_\alpha f(x)-T_\alpha f(r_\alpha.x)\} \\
=&\sum_{\alpha\in\mathcal{R}^+}\frac{k_\alpha}{2}\coth\frac{(\alpha,x)}{2}\{\partial_\alpha f(x)-\partial_\alpha f(r_\alpha.x)\} \\
+&\sum_{\alpha,\beta\in\mathcal{R}^+}\frac{k_\alpha k_\beta}{4}(\alpha,\beta)\coth\frac{(\alpha,x)}{2}\coth\frac{(\beta,x)}{2}\{f(x)-f(r_\alpha.x)\} \\
-&\sum_{\alpha,\underbrace{\beta}_{\beta'}\in\mathcal{R}^+}\frac{k_\alpha\overbrace{k_\beta}^{k'_\beta}}{4}\overbrace{(\alpha,\beta)}^{-(\alpha,\beta')}\coth\frac{(\alpha,x)}{2}\coth\frac{\overbrace{(\beta,r_\alpha.x)}^{(\beta',x)}}{2}\{f(r_\alpha.x)-f(\overbrace{r_\beta r_\alpha}^{r_\alpha r_{\beta'}}.x)\} \\
-&\sum_{\alpha,\beta\in\mathcal{R}^+}\frac{k_\alpha k_\beta}{4}(\alpha,\beta)\coth\frac{(\alpha,x)}{2}f(r_\beta.x) \\
+&\sum_{\alpha,\beta\in\mathcal{R}^+}\frac{k_\alpha k_\beta}{4}(\alpha,\beta)\coth\frac{(\alpha,x)}{2}f(r_\beta r_\alpha.x) \\
=&\sum_{\alpha\in\mathcal{R}^+}\frac{k_\alpha}{2}\coth\frac{(\alpha,x)}{2}\{\partial_\alpha f(x)-\partial_\alpha f(r_\alpha.x)\} \\
+&\sum_{\alpha,\beta\in\mathcal{R}^+}\frac{k_\alpha k_\beta}{4}(\alpha,\beta)\coth\frac{(\alpha,x)}{2}\coth\frac{(\beta,x)}{2}\{f(x)-f(r_\beta r_\alpha.x)\} \\
-&\sum_{\alpha,\beta\in\mathcal{R}^+}\frac{k_\alpha k_\beta}{4}(\alpha,\beta)\coth\frac{(\alpha,x)}{2}\{f(r_\beta.x)-f(r_\beta r_\alpha.x)\}.
\end{aligned}$$

Thanks to the following lemma, we can remove the hyperbolic cotangent in the second sum.

Lemma 6.1 *Let $\mathcal{R}$ be an integral root system (non necessarily reduced). Then*

$$\sum_{\alpha,\beta\in\mathcal{R}^+,r_\beta\circ r_\alpha=\tau} k_\alpha k_\beta(\alpha,\beta)\{\coth\frac{(\alpha,x)}{2}\coth\frac{(\beta,x)}{2}-1\}=0$$

for all non trivial rotation τ.

Proof of the lemma: Applying the Euclidean Laplacian to the Weyl denominator formula

$$\prod_{\alpha\in\mathcal{R}^+}\{e^{\frac{(\alpha,x)}{2}}-e^{-\frac{(\alpha,x)}{2}}\}=\sum_{w\in W}e^{(w.\rho,x)}$$

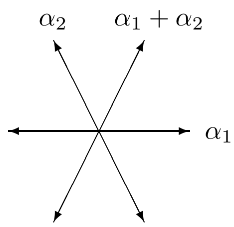
we get the identity

$$\sum_{\alpha,\beta\in\mathcal{R}^+,\alpha\neq\beta}(\alpha,\beta)\{\coth\frac{(\alpha,x)}{2}\coth\frac{(\beta,x)}{2}-1\}=0$$

which holds for all reduced root system. Now by restricting to the different root systems of rank 2, we see that this relation is equivalent to the lemma.

• $\mathbf{A_1}\times\mathbf{A_1}$**:** trivial.

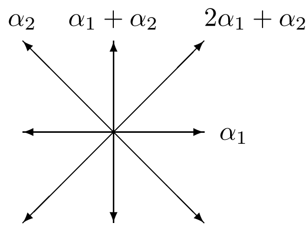
• $\mathbf{A_2}$**:**



The lemma reduces to the identity

$$\frac{k^2}{2}\{-\coth\frac{\alpha_1}{2}\coth\frac{\alpha_2}{2}+\coth\frac{\alpha_1}{2}\coth\frac{\alpha_1+\alpha_2}{2}+\coth\frac{\alpha_2}{2}\coth\frac{\alpha_1+\alpha_2}{2}-1\}=0.$$

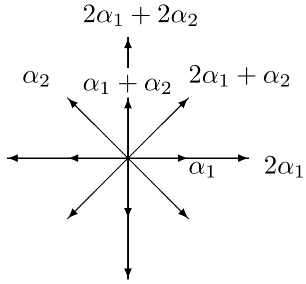
• $\mathbf{B_2}=\mathbf{C_2}$**:**



The lemma reduces to the identity

$$\begin{aligned}k_1k_2\{ &- \coth\frac{\alpha_1}{2}\coth\frac{\alpha_2}{2}+\coth\frac{\alpha_1}{2}\coth\frac{2\alpha_1+\alpha_2}{2}\\ &+ \coth\frac{\alpha_2}{2}\coth\frac{\alpha_1+\alpha_2}{2}+\coth\frac{\alpha_1+\alpha_2}{2}\coth\frac{2\alpha_1+\alpha_2}{2}-2\}=0.\end{aligned}$$

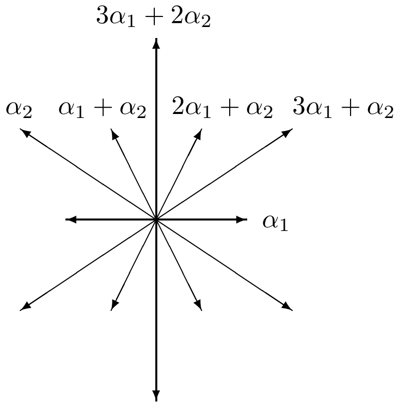
• $\mathbf{BC_2}$**:**



The lemma reduces to the following identities of type $B_2 = C_2$

$$
\begin{aligned}
k_1k_2\{ \quad - &\quad \coth\frac{\alpha_1}{2}\coth\frac{\alpha_2}{2} + \coth\frac{\alpha_1}{2}\coth\frac{2\alpha_1+\alpha_2}{2} \\
+ &\quad \coth\frac{\alpha_2}{2}\coth\frac{\alpha_1+\alpha_2}{2} + \coth\frac{\alpha_1+\alpha_2}{2}\coth\frac{2\alpha_1+\alpha_2}{2} - 2\} = 0 \\
2k_2k_3\{ \quad - &\quad \coth\frac{\alpha_1}{2}\coth\frac{\alpha_2}{2} + \coth\frac{\alpha_1}{2}\coth\frac{2\alpha_1+\alpha_2}{2} \\
+ &\quad \coth\frac{\alpha_2}{2}\coth\frac{\alpha_1+\alpha_2}{2} + \coth\frac{\alpha_1+\alpha_2}{2}\coth\frac{2\alpha_1+\alpha_2}{2} - 2\} = 0.
\end{aligned}
$$

• $\mathbf{G_2}$:



The lemma reduces to the following identities, the last ones being of type A_2

$$\begin{aligned}
\frac{3k_1k_2}{2}\{ \quad - \quad & \coth\frac{\alpha_1}{2}\coth\frac{\alpha_2}{2}+\coth\frac{\alpha_1}{2}\coth\frac{3\alpha_1+\alpha_2}{2} \\
+ \quad & \coth\frac{\alpha_2}{2}\coth\frac{\alpha_1+\alpha_2}{2}+\coth\frac{\alpha_1+\alpha_2}{2}\coth\frac{3\alpha_1+2\alpha_2}{2} \\
+ \quad & \coth\frac{2\alpha_1+\alpha_2}{2}\coth\frac{3\alpha_1+\alpha_2}{2}+\coth\frac{2\alpha_1+\alpha_2}{2}\coth\frac{3\alpha_1+2\alpha_2}{2}-4\}=0 \\
\frac{k_1^2}{2}\{ \quad - \quad & \coth\frac{\alpha_1}{2}\coth\frac{\alpha_1+\alpha_2}{2}+\coth\frac{\alpha_1}{2}\coth\frac{2\alpha_1+\alpha_2}{2} \\
+ \quad & \coth\frac{\alpha_1+\alpha_2}{2}\coth\frac{2\alpha_1+\alpha_2}{2}-1\}=0 \\
\frac{3k_2^2}{2}\{ \quad - \quad & \coth\frac{\alpha_2}{2}\coth\frac{3\alpha_1+\alpha_2}{2}+\coth\frac{\alpha_2}{2}\coth\frac{3\alpha_1+2\alpha_2}{2} \\
+ \quad & \coth\frac{3\alpha_1+\alpha_2}{2}\coth\frac{3\alpha_1+2\alpha_2}{2}-1\}=0.
\end{aligned}$$

■

Eventually we get the expression of the Heckman-Opdam Laplacian:

$$
\begin{aligned}
\mathcal{L}f(x) &= \sum_{j=1}^{n} T_{\xi_j}^2 f(x) \\
&= \sum_{j=1}^{n} \partial_{\xi_j} T_{\xi_j} f(x) + \sum_{\alpha\in\mathcal{R}^+} \frac{k_\alpha}{2} \coth\frac{(\alpha,x)}{2} \overbrace{\sum_{j=1}^{n} (\alpha,\xi_j)\{T_{\xi_j} f(x) - T_{\xi_j} f(r_\alpha.x)\}}^{T_\alpha f(x) - T_\alpha f(r_\alpha.x)} \\
&- \sum_{\alpha\in\mathcal{R}^+} \frac{k_\alpha}{2} \underbrace{\sum_{j=1}^{n} (\alpha,\xi_j) T_{\xi_j} f(r_\alpha.x)}_{T_\alpha f(r_\alpha.x)} \\
&= \Delta f(x) + \sum_{\beta\in\mathcal{R}^+} \frac{k_\beta}{4} \underbrace{\sum_{j=1}^{n} (\beta,\xi_j)^2}_{|\beta|^2} \overbrace{(1-\coth^2\frac{(\beta,x)}{2})}^{-\sinh^{-2}\frac{(\beta,x)}{2}} \{f(x) - f(r_\beta.x)\} \\
&+ \sum_{\beta\in\mathcal{R}^+} \frac{k_\beta}{2} \coth\frac{(\beta,x)}{2} \overbrace{\sum_{j=1}^{n} (\beta,\xi_j)\{\partial_{\xi_j} f(x) - \partial_{r_\beta.\xi_j} f(r_\beta.x)\}}^{\partial_\beta f(x) + \partial_\beta f(r_\beta.x)} \\
&- \sum_{\beta\in\mathcal{R}^+} \frac{k_\beta}{2} \underbrace{\sum_{j=1}^{n} (\beta,\xi_j) \partial_{r_\beta.\xi_j} f(r_\beta.x)}_{-\partial_\beta f(r_\beta.x)} \\
&+ \sum_{\alpha\in\mathcal{R}^+} \frac{k_\alpha}{2} \coth\frac{(\alpha,x)}{2} \{\partial_\alpha f(x) - \partial_\alpha f(r_\alpha.x)\} \\
&+ \sum_{\alpha,\beta\in\mathcal{R}^+} \frac{k_\alpha k_\beta}{4} (\alpha,\beta)\{f(x) - f(r_\beta r_\alpha.x)\} \\
&- \sum_{\alpha,\beta\in\mathcal{R}^+} \frac{k_\alpha k_\beta}{4} (\alpha,\beta) \coth\frac{(\alpha,x)}{2} \{f(r_\beta.x) - f(r_\beta r_\alpha.x)\} \\
&- \sum_{\alpha,\underbrace{\beta}_{\beta'}\in\mathcal{R}^+} \frac{k_\alpha \overbrace{k_\beta}^{k'_\beta}}{4} \overbrace{(\alpha,\beta)}^{-(\alpha,\beta')} \coth\frac{\overbrace{(\beta, r_\alpha.x)}^{(\beta',x)}}{2} \{f(r_\alpha.x) - f(\overbrace{r_\beta r_\alpha}^{r_\alpha r_{\beta'}}.x))\} \\
&+ \sum_{\alpha,\beta\in\mathcal{R}^+} \frac{k_\alpha k_\beta}{4} (\alpha,\beta) f(r_\beta r_\alpha.x) \\
&= \Delta f(x) + \sum_{\alpha\in\mathcal{R}^+} k_\alpha \coth\frac{(\alpha,x)}{2} \partial_\alpha f(x) + |\rho|^2 f(x) \\
&- \sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{|\alpha|^2}{4\sinh^2\frac{(\alpha,x)}{2}} \{f(x) - f(r_\alpha.x)\}.
\end{aligned}
$$

References

[1] **Anker J-Ph.:** *La forme exacte de l'estimatation fondamentale de Harish-Chandra*, C. R. Acad. Sci. Paris Série I Math. 305 (1987), 371 – 374.

[2] **Anker J-Ph.:** *The spherical Fourier transform of rapidly decreasing functions. A simple proof of a characterization due to Harish-Chandra, Helgason, Trombi, and Varadarajan*, J. Funct. Anal. 96 (1991), 331 - 349.

[3] **Anker J-Ph., Bougerol Ph., Jeulin T.:** *The infinite Brownian loop on a symmetric space*, Rev. Mat. Iberoamericana 18 (2002), 41 - 97.

[4] **Anker J-Ph., Ji L.:** *Heat kernel and Green function estimates on noncompact symmetric spaces*, Geom. Funct. Anal. 9 (1999), 1035 – 1091.

[5] **Anker J-Ph., Ostellari P.:** *The heat kernel on noncompact symmetric spaces*, in *Lie groups and symmetric spaces : In memory of Karpelevich F.I.*, Gindikin S.G. (ed.), Amer. Math. Soc. Transl. 210 (2003), 27 - 46.

[6] **Cherednik I.:** *A unification of Knizhnik-Zamolodchnikov equations and Dunkl operators via affine Hecke algebras*, Invent. Math. 106 (1991), 411 - 432.

[7] **Cherednik I.:** *Inverse Harish-Chandra transform and difference operators*, Internat. Math. Res. Notices 15 (1997), 733 - 750.

[8] **Delorme P.:** *Transformation de Fourier hypergeometrique*, J. Funct. Anal. 168 (1999), 239 - 312.

[9] **Ethier N., Kurtz G.:** *Markov processes. Characterization and convergence*, Wiley Series in Probab. Math. Stat. (1986).

[10] **Helgason S.:** *Groups and Geometric Analysis*, Academic Press (1984).

[11] **Heckman G. J., Opdam E. M.:** *Root systems and hypergeometric functions I.* Compositio Math. 64 (1987), 329 – 352.

[12] **Heckman G. J., Schlichtkrull H.:** *Harmonic analysis and special functions on symmetric spaces*, Academic Press (1994).

[13] **De Jeu M.F.E.:** *The Dunkl transform*, Invent. Math. 113 (1993), 147 – 162.

[14] **Opdam E. M.:** *Harmonic analysis for certain representations of graded Hecke algebras*, Acta Math. 175 (1995), 75 – 121.

[15] **Protter M.H., Weinberger H.F.:** *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, 1967

[16] **Revuz D., Yor M.:** *Continuous martingales and Brownian motion*, Springer-Verlag, third ed. (1999).

[17] **Rösler M.:** *Generalized Hermite polynomials and the heat equation for Dunkl operators*, Comm. Math. Phys. 192, (1998), 519 − 542.

[18] **Sawyer P.:** *A global estimate of the Legendre function for the root systems of type A with arbitrary multiplicities*, to appear in Canad. Math. Bull.

[19] **Schapira Br.:** *The Heckman-Opdam Markov processes*, available on arxiv math.PR/0605020.